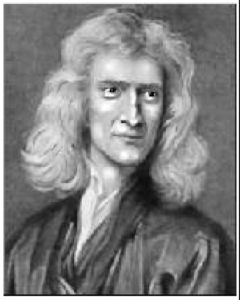
MATH 21a REVIEW, 5/14/2003 **Math 21a, O. Knill**

Your lucky BINGO numbers are below. During the presentation, we will pick randomly 5 of 9 mathematicians relevant for multivariable calculus. If one of your mathematicians appear, cross it off. There is a big change that one student gets all 5 right. If you have collected 4 mathematicians, raise your voice.



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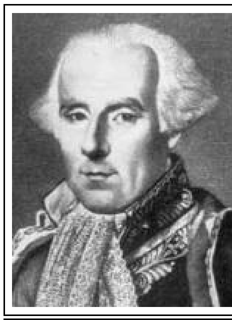

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1. Geometry of Space

- coordinates and vectors in the plane and in space
- $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- dot product $v.w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos(\alpha)$
- cross product, $v.(v \times w) = 0$, $w.(v \times w) = 0$, $|v \times w| = |v||w|\sin(\alpha)$
- triple cross product $u \cdot (v \times w)$ volume of parallelepiped
- parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$
- scalar projection $\text{comp}_w(v) = v \cdot w/|w|$, vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- completion of square technique: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x-2)^2 + y^2 = -3$
- distance $d(P, Q) = |\vec{PQ}| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$

2. Lines, Planes, Functions

- symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{z-z_0}{c}$
- plane $ax + by + cz = d$
- parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- switch from parametric to implicit descriptions for lines and planes
- domain and range of functions $f(x, y)$
- graph $G = \{(x, y, f(x, y))\}$
- intercepts: intersections of G with coordinate axes
- traces: intersections with coordinate planes
- generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, parabolic hyperboloid
- plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
- line $\frac{(x-x_0)}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $\vec{v} = (a, b, c)$
- sets $g(x, y, z) = c$ describe surfaces, example graphs $g(x, y, z) = z - f(x, y)$
- linear equation $2x + 3y + 5z = 7$) defines plane
- quadratic equation i.e. $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
- distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
- distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$
- distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
- finding plane through three points P, Q, R: find first normal vector

3. Curves

- plane and space curves $\vec{r}(t)$
- velocity $\vec{r}'(t)$, Acceleration $\vec{r}\,''(t)$
- unit tangent vector $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$
- unit normal vector $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$
- binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
- curvature $\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)|$
- arc length $\int_a^b |\vec{r}\,'(t)|\, dt$
- $\vec{r}\,'(t)$ is tangent to the curve
- $\vec{v} = \vec{r}\,'$ then $\vec{r} = \int_0^t \vec{v}\, dt + \vec{c}$
- $\kappa(t) = \frac{|r\,'(t) \times r\,''(t)|}{|r\,'(t)|^3}$
- $\frac{d}{dt}(\vec{v}(t) \cdot \vec{w}(t)) = \vec{v}'(t) \cdot \vec{w}(t) + \vec{v}(t) \cdot \vec{w}'(t)$
- T, N, B are unit vectors which are perpendicular to each other
- find parameterizations of basic curves (i.e. intersections of surfaces)

4. Surfaces

- polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$
- cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$
- spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
- $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
- $g(r, \theta, z) = 0$ cylindrical surface, especially $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
- $g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
- $f(x, y) = c$ level curves of $f(x, y)$
- $g(x, y, z) = c$ level surfaces of $g(x, y, z)$
- circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = (r\cos t, r\sin t)$.
- ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = (a\cos t, b\sin t)$
- sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u, v) = (r\cos u\sin v, r\sin u\sin v, r\cos v)$
- ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u, v) = (a\cos u\sin v, b\sin u\sin v, c\cos v)$
- line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
- plane: $ax + by + cz = d$, $\vec{r}(u, v) = \vec{r}_0 + u\vec{v} + v\vec{w}, (a, b, c) = \vec{v} \times \vec{w}$
- surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u, v) = (f(v)\cos(u), f(v)\sin(u), v)$
- graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(u, v) = (u, v, f(u, v))$

5. Partial Derivatives

- $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
- partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ 1D wave equation
- $f_x - f_t = 0$ transport equation
- $f_x f - f_t = 0$ Burger equation
- $f_{xx} + f_{yy} = 0$ Laplace equation
- $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
- tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
- estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
- $f(x, y)$ differentiable if f_x, f_y are continuous
- $f_{xy} = f_{yx}$ Clairot's theorem
- $\vec{r}_u(u, v)$, $\vec{r}_v$ tangent to surface $\vec{r}(u, v)$

6. Chain Rule

$\nabla f(x,y) = (f_x, f_y)$, $\nabla f(x,y,z) = (f_x, f_y, f_z)$, gradient
$D_v f = \nabla i f \cdot v$ directional derivative
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ chain rule
$\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ which contains (x_0, y_0, z_0).
$\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$ normal line to surface $f(x,y,z) = c$ at (x_0, y_0, z_0)
$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane at (x_0, y_0, z_0)
directional derivative is maximal in the $\vec{v} = \nabla f$ direction
$f(x,y)$ increases, if we walk on the xy-plane in the ∇f direction
partial derivatives are special directional derivatives
if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$
implicit differentiation: $f(x, y(x)) = 0$, $f_x 1 + f_y y'(x) = 0$ gives $y'(x) = -f_x/f_y$

7. Extrema

$\nabla f(x,y) = (0,0)$, critical point or stationary point
$D = f_{xx} f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
$f(x_0, y_0) \geq f(x,y)$ in a neighborhood of (x_0, y_0) local maximum
$f(x_0, y_0) \leq f(x,y)$ in a neighborhood of (x_0, y_0) local minimum
$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange multiplier
two constraints: $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$
Second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ local max, $\nabla f = (0,0), D > 0, f_{xx} > 0$ local min, $\nabla f = (0,0), D < 0$ saddle

8. Double Integrals

$\int\int_R f(x,y)\; dA$ double integral
$\int_a^b \int_c^d f(x,y)\; dydx$ integral over rectangle
$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dydx$ type I region
$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\; dxdy$ type II region
$\int\int_R f(r,\theta) r\; dr d\theta$ polar coordinates
$\int\int_R |\vec{r}_u \times \vec{r}_v|\; dudv$ surface area
$\int_a^b \int_c^d f(x,y)\; dydx = \int_c^d \int_a^b f(x,y)\; dxdy$ Fubini
$\int\int_R 1\; dxdy$ area of region R
$\int\int_R f(x,y)\; dxdy$ volume of solid bounded by graph(f) xy-plane

9. Triple Integrals

$\int\int\int_R f(x,y,z)\; dV$ triple integral
$\int_a^b \int_c^d \int_u^v f(x,y,z)\; dydx$ integral over rectangular box
$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\; dzdydx$ type I region
$f(r,\theta,z)\ \boxed{r}\ dzdrd\theta$ cylindrical coordinates
$\int\int\int_R f(\rho,\theta,z)\ \boxed{\rho^2 \sin(\phi)}\ dzdrd\theta$ spherical coordinates
$\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ Jacobian of $T(u,v) = (x(u,v), y(u,v))$
$\frac{\partial(x,y,z)}{(u,v,w)} = [\nabla x, \nabla y, \nabla z]$ Jacobian of $T(u,v,w) = (x,y,z)$
$\int_a^b \int_c^d \int_u^v f(x,y,z)\; dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\; dxdydz$ Fubini
$V = \int\int\int_R 1\; dV$ volume of solid R
$M = \int\int\int_R \rho(x,y,z)\; dV$ mass of solid R with density ρ
$(\int\int\int_R x\; dV/V, \int\int\int_R y\; dV/V, \int\int\int_R z\; dV/V)$ center of mass

10. Line Integrals

$F(x,y) = (P(x,y), Q(x,y))$ vector field in the plane
$F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ vector field in space
$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\ dt$ line integral
$F(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative
f is the potential of F
Fundamental thm of line int: $F(x,y) = \nabla f(x,y)$, $\int_a^b F(r(t)) \cdot r'(t))\ dt = f(r(b)) - f(r(a))$
For smooth gradient fields in simply connected region R $\int_C F\ dr = 0$, for all closed curves C

11. Green's Theorem

$F(x,y) = (P,Q)$, $\text{curl}(F) = Q_x - P_y = \nabla \times F$. $\text{div}(F) = P_x + Q_y = \nabla \cdot F$
$F(x,y,z) = (P,Q,R)$, $\text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F$, $\text{div}(P,Q,R) = P_x + Q_y + R_z = \nabla \cdot F$
$\Delta f = \text{divgrad}(f) = f_{xx} + f_{yy} + f_{zz}$ Laplacian for functions
$\Delta F = \Delta(P,Q,R) = (\Delta P, \Delta Q, \Delta R)$ Laplacian for vector fields
$\nabla = (\partial_x, \partial_y, \partial_z)$, $\text{grad}(F) = \nabla f$,$\text{curl}(F) = \nabla \times F$, $\text{div}(F) = \nabla \cdot F$
$\Delta f = \nabla \cdot \nabla f$
Green's theorem: C boundary of R, then $\int_C F \cdot dr = \int\int_R \text{curl}(F)\ dxdy$
$\text{div}(\text{curl}(F)) = 0$
$\text{curl}(\text{grad}(F)) = \vec{0}$
$\text{curl}(\text{curl}(F)) = \text{grad}(\text{div}(F)) - \Delta(F)$

12. Stokes and Divergence Theorem

$F(x,y,z)$ vector field, $S = r(R)$ parametrized surface
$r_u \times r_v$ normal vector, $\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$ unit normal vector
$|r_u \times r_v| dudv = dS$ surface element
$r_u \times r_v dudv = d\vec{S} = \vec{n} dS$ normal surface element
$\int\int_S f\ dS = \int\int_S f(r(u,v)) |r_u \times r_v|\ dudv$ surface integral
$\int\int_S F\ d\vec{S} = \int\int_S F(r(u,v)) \cdot (r_u \times r_v)\ dudv$ flux integral
Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \int\int_S \text{curl}(F) \cdot dS$
Divergence theorem: E bounded by S then $\int\int_S F \cdot dS = \int\int\int_E \text{div}(F)\ dV$