34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the

calculated area is about $dA = 6(0.002) + \tfrac{5}{2}(0.002) = 0.017$ m^2 or 170 cm^2.

(b) $z = \sqrt{x^2 + y^2}$, $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum

error in the calculated hypotenuse length is about $dz = \tfrac{5}{13}(0.002) + \tfrac{12}{13}(0.002) = \tfrac{0.17}{65} \approx 0.0026$ m or
0.26 cm.

35. $\dfrac{dw}{dt} = \dfrac{1}{2\sqrt{x}}\left(2e^{2t}\right) + \dfrac{2y}{z}\left(3t^2 + 4\right) + \dfrac{-y^2}{z^2}(2t) = e^t + \dfrac{2y}{z}\left(3t^2 + 4\right) - 2t\dfrac{y^2}{z^2}$

36. $\dfrac{\partial z}{\partial u} = (-y\sin xy - y\sin x)(2u) + (-x\sin xy + \cos x) = \cos x - 2uy\sin x - (\sin xy)(x + 2uy)$,

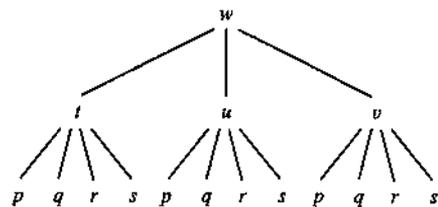$\dfrac{\partial z}{\partial v} = (-y\sin xy - y\sin x)(1) + (-x\sin xy + \cos x)(-2v) = -2v\cos x + (\sin xy)(2vx - y) - y\sin x$

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1, 2) = 3$ and $y = h(1, 2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3, 6)g_s(1, 2) + f_y(3, 6)\,h_s(1, 2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3, 6)g_t(1, 2) + f_y(3, 6)\,h_t(1, 2) = (7)(4) + (8)(10) = 108$.

38.

Using the tree diagram as a guide, we have

$\dfrac{\partial w}{\partial p} = \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial p} + \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial p} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial p}$

$\dfrac{\partial w}{\partial q} = \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial q} + \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial q} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial q}$

$\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial r} + \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial r} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial r}$

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial t}\dfrac{\partial t}{\partial s} + \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial w}{\partial v}\dfrac{\partial v}{\partial s}$

39. $\dfrac{\partial z}{\partial x} = 2xf'\left(x^2 - y^2\right)$, $\dfrac{\partial z}{\partial y} = 1 - 2yf'\left(x^2 - y^2\right)$ $\left[\text{where } f' = \dfrac{df}{d\left(x^2 - y^2\right)}\right]$. Then

$y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 2xyf'\left(x^2 - y^2\right) + x - 2xyf'\left(x^2 - y^2\right) = x$.

40. $A = \tfrac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(y\sin\theta)\dfrac{dx}{dt} + (x\sin\theta)\dfrac{dy}{dt} + (xy\cos\theta)\dfrac{d\theta}{dt}\right]$. So when $x = 40$, $y = 50$ and $\theta = \tfrac{\pi}{6}$,

$\dfrac{dA}{dt} = \dfrac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \dfrac{35 + 50\sqrt{3}}{2} \approx 60.8$ in^2/s.

41. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\, y + \dfrac{\partial z}{\partial v}\dfrac{-y}{x^2}$ and

$$\frac{\partial^2 z}{\partial x^2} = y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}y + \frac{\partial^2 z}{\partial v\partial u}\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\partial v}y\right)$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}$$

Also $\dfrac{\partial z}{\partial y} = x\,\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\partial v}x\right)$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Thus

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = \frac{2y}{x}\frac{\partial z}{\partial v} + x^2 y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2 y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}$$

$$= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\frac{\partial^2 z}{\partial u\partial v} = 2v\frac{\partial z}{\partial v} - 4uv\frac{\partial^2 z}{\partial u\partial v}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

42. $F(x,y,z) = e^{xyz} - yz^4 - x^2 z^3 = 0$, so $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yze^{xyz} - 2xz^3}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \dfrac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}$

and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xze^{xyz} - z^4}{xye^{xyz} - 4yz^3 - 3x^2 z^2} = \dfrac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2 z^2}$.

43. $\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}},\ \dfrac{xz^2 e^{x\sqrt{y}}}{2\sqrt{y}},\ 2ze^{x\sqrt{y}}\right\rangle = ze^{x\sqrt{y}}\left\langle z\sqrt{y},\ \dfrac{xz}{2\sqrt{y}},\ 2\right\rangle$

44. (a) By Theorem 11.6.15, the maximum value of the directional derivative occurs when **u** has the same direction as the gradient vector.

(b) It is a minimum when **u** is in the direction opposite to that of the gradient vector (that is, **u** is in the direction of $-\nabla f$), since $D_{\mathbf{u}}f = |\nabla f|\cos\theta$ (see the proof of Theorem 11.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when **u** is perpendicular to the gradient vector, since then $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}f = |\nabla f|\cos\theta = \tfrac{1}{2}|\nabla f| \iff$
   $\cos\theta = \tfrac{1}{2} \iff \theta = \tfrac{\pi}{3}$.

45. $\nabla f = \langle 1/\sqrt{x},\ -2y\rangle$, $\nabla f(1,5) = \langle 1,\ -10\rangle$, $\mathbf{u} = \tfrac{1}{5}\langle 3,\ -4\rangle$. Then $D_{\mathbf{u}}f(1,5) = \tfrac{43}{5}$.

46. $\nabla f = \langle 2xy + \sqrt{1+z},\ x^2,\ x/(2\sqrt{1+z})\rangle$, $\nabla f(1,2,3) = \langle 6,1,\tfrac{1}{4}\rangle$, and $\mathbf{u} = \langle\tfrac{2}{3},\tfrac{1}{3},-\tfrac{2}{3}\rangle$. Then
   $D_{\mathbf{u}}f(1,2,3) = \tfrac{25}{6}$.

47. $\nabla f = \langle 2xy,\ x^2 + 1/(2\sqrt{y})\rangle$, $|\nabla f(2,1)| = |\langle 4,\tfrac{9}{2}\rangle|$. Thus the maximum rate of change of f at $(2,1)$ is $\dfrac{\sqrt{145}}{2}$ in
   the direction $\langle 4,\tfrac{9}{2}\rangle$.

48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy} \rangle$, $\nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle$ is the direction of most rapid increase while the rate is $|\langle 2, 0, 1 \rangle| = \sqrt{5}$.

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

**50.** The surfaces are $f(x, y, z) = z - 2x^2 + y^2 = 0$ and $g(x, y, z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2, 2, 4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line.
$\nabla f(x, y, z) = \langle -4x, 2y, 1 \rangle \quad \Rightarrow \quad \nabla f(-2, 2, 4) = \langle 8, 4, 1 \rangle$, $\nabla g(x, y, z) = \langle 0, 0, 1 \rangle \quad \Rightarrow$

$\nabla g\langle -2, 2, 4 \rangle = \langle 0, 0, 1 \rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\mathbf{i} - 8\mathbf{j}$. Thus, parametric equations are:

$x = -2 + 4t$, $y = 2 - 8t$, and $z = 4$.

51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10 \quad \Rightarrow \quad f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4, 1) > 0$, $D(-4, 1) = 3 > 0$, so $f(-4, 1) = -11$ is a local minimum.
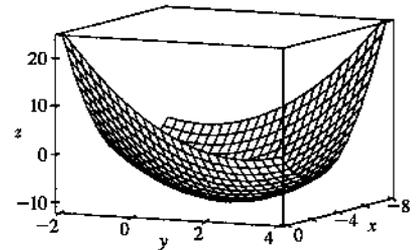

52. $f(x, y) = x^3 - 6xy + 8y^3 \quad \Rightarrow \quad f_x = 3x^2 - 6y$, $f_y = -6x + 24y^2$, $f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0, 0)$, $(1, \frac{1}{2})$. $D(0, 0) = -36 < 0$ so $(0, 0)$ is a saddle point while $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and $D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local minimum.
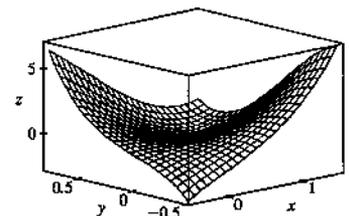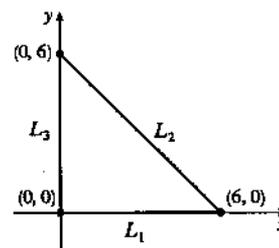

53. $f(x, y) = 3xy - x^2y - xy^2 \quad \Rightarrow \quad f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(1, 1)$. $D(0, 0) = D(3, 0) = D(0, 3) = -9 < 0$ so $(0, 0)$, $(3, 0)$, and $(0, 3)$ are saddle points. $D(1, 1) = 3 > 0$ and $f_{xx}(1, 1) = -2 < 0$, so $f(1, 1) = 1$ is a local maximum.
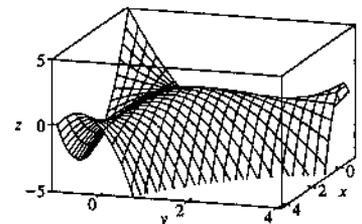
54. $f(x,y) = \left(x^2 + y\right) e^{y/2}$ $\Rightarrow$ $f_x = 2xe^{y/2}$, $f_y = e^{y/2}\left(2 + x^2 + y\right)/2$, $f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}\left(4 + x^2 + y\right)/4$, $f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0,-2) > 0$, $D(0,-2) = e^{-2} - 0 > 0$ so $f(0,-2) = -2/e$ is a local minimum.
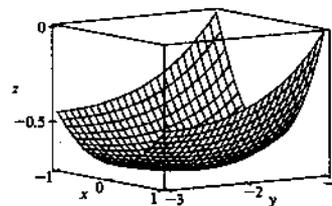
55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D. On L_1, $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2, $x = -y + 6$ and $f9 - y + 6, y) = y^2(6 - y)(-2) = -2\left(6y^2 - y^3\right)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3, $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

56. Inside D: $f_x = 2xe^{-x^2-y^2}\left(1 - x^2 - 2y^2\right) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}\left(2 - x^2 - 2y^2\right) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f((\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}\left(4 + y^2\right)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0,\pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$
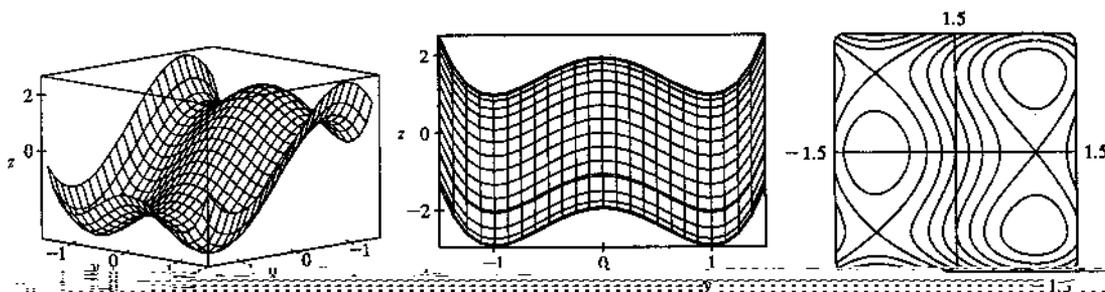
From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minima $f(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \iff x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \iff y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1, 0) = 24 > 0$ and $f_{xx}(-1, 0) = -6 < 0$ indicating a local maximum $f(-1, 0) = 2$; $D(1, \pm 1) = 48 > 0$ and $f_{xx}(1, \pm 1) = 6 > 0$ indicating local minima $f(1, \pm 1) = -3$; and $D(-1, \pm 1) = -48$ and $D(1, 0) = -24$, indicating saddle points.

58. $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4 \implies f_x(x, y) = -4x - 8y$, $f_y(x, y) = 10 - 8x - 4y^3$. Now $f_x(x, y) = 0 \implies x = -2y$, and substituting this into $f_y(x, y) = 0$ gives $10 + 16y - 4y^3 = 0 \iff 5 + 8y - 2y^3 = 0$.
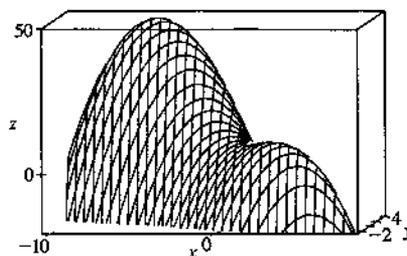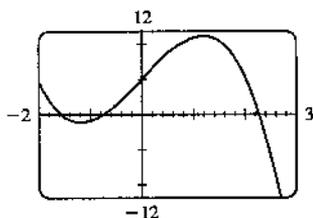
From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x, y) = x^2 y$, $g(x, y) = x^2 + y^2 = 1 \implies \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm \frac{1}{\sqrt{3}}$ and $x = \pm \sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0, 0) = 0$.] Thus the possible points are $\left(\pm \sqrt{\frac{2}{3}}, \pm \frac{1}{\sqrt{3}} \right)$ and the absolute maxima are $f\left(\pm \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}} \right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}} \right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x, y) = 1/x + 1/y$, $g(x, y) = 1/x^2 + 1/y^2 = 1 \implies \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^{-3}$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm \sqrt{2}$ and the possible points are $(\pm \sqrt{2}, \pm \sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

61. $f(x, y, z) = xyz$, $g(x, y, z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g$ $\Rightarrow$ $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x,

y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z}$ $\Rightarrow$

$2y^2z = 2x^2z$ $\Rightarrow$ $y^2 = x^2$, and similarly $2yz^2 = 2x^2y$ $\Rightarrow$ $z^2 = x^2$. Substituting into the constraint equation

gives $x^2 + x^2 + x^2 = 3$ $\Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus the possible points are

$(1, 1, \pm 1)$, $(1, -1, \pm 1)$, $(-1, 1, \pm 1)$, $(-1, -1, \pm 1)$. The absolute maximum is

$f(1, 1, 1) = f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = 1$ and the absolute minimum is

$f(1, 1, -1) = f(1, -1, 1) = f(-1, 1, 1) = f(-1, -1, -1) = -1$.

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = x - y + 2z = 2$ $\Rightarrow$

$\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and (1) $2x = \lambda + \mu$, (2) $4y = \lambda - \mu$,

(3) $6z = \lambda + 2\mu$, (4) $x + y + z = 1$, (5) $x - y + 2z = 2$. Then six times (1) plus three times (2) plus two times

(3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus

...3. Solving ...λ + (μ.... ...

(2) and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$,

1) satisfies both constraints and

e is no absolute maximum.

..., $x \neq 0$, $y \neq 0$ and $z \neq 0$, so

... $\frac{}{z^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so

$= \pm \frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z

$\frac{2}{3}z^2)z^3 = 2$ or $z = \pm 3^{1/4}$ and the

$^{/4}$). However at each of these points

$f(2, 1, 1) = 6 > 2\sqrt{3}$. Thus f has

constraint $xy^2z^3 = 2$.

$f(x, z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then

$f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies

...four times (3) implies $12(x - y - 2z) = 7\lambda - 11\mu$ so (5) gives $7\lambda - 11\mu = ...$

$7\lambda + 11\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (

$z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0, 0, $

$f(0, 0, 1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and ther

63. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = xy^2z^3 = 2$ $\Rightarrow$

$\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$

(1) $2x = \lambda y^2z^3$, (2) $1 = \lambda xz^3$, (3) $2 = 3\lambda xy^2z$. Then (2) and (3) imply $\frac{}{x}$

$y = \pm z\sqrt{\frac{2}{3}}$. Similarly (1) and (3) imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x =$

must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x, y, z) = 2$ implies $\frac{1}{\sqrt{3}}z$

possible points are $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$, $\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1}\right)$

f takes on the same value, $2\sqrt{3}$. But $(2, 1, 1)$ also satisfies $g(x, y, z) = 2$ an

an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the

Alternate solution: $g(x, y, z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize

$f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and

$2x^2z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_z = 0$ implies $-6x + 2x = 0$ of $x = \frac{1}{\sqrt[4]{3}}$. So the two critical

points are $\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right)$. Then $D\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right) = (2 + 4)\left(2 + \frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and

$f_{xx}\left(\pm \frac{1}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the

origin are $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm \sqrt[4]{3}\right)$.

64. $V = xyz$, say x is the length and $x + 2y + 2z \le 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to

$x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$ and

$xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is

$V = 11{,}664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x, y, z) = 108$ and $V(104, 1, 1) = 104$ cubic units,

$(36, 18, 18)$ gives an absolute maximum of V subject to $g(x, y, z) = 108$. But if $x + 2y + 2z < 108$, there exists

$\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108 - \alpha)/6 = z$,

$x = (108 - \alpha)/3$ with $V = (108 - \alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11{,}664$. Hence we have shown that the

maximum of V subject to $g(x, y, z) \le 108$ is the maximum of V subject to $g(x, y, z) = 108$ (an intuitively obvious

fact).

65.
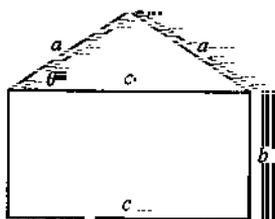
The area of the triangle is $\frac{1}{2}ca \sin \theta$ and the area of the rectangle is cb.

Thus, the area of the whole object is $f(a, b, c) = \frac{1}{2}ca \sin \theta + bc$. The

perimeter of the object is $g(a, b, c) = 2a + 2b + c = P$. To simplify $\sin \theta$

in terms of a, b, and c notice that $a^2 \sin^2 \theta + \left(\frac{1}{2}c\right)^2 = a^2 \Rightarrow$

$\sin \theta = \frac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a, b, c) = \frac{c}{4}\sqrt{4a^2 - c^2} + bc$.

(Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must

find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: (1) $ca(4a^2 - c^2)^{-1/2} = 2\lambda$,

(2) $c = 2\lambda$, (3) $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$, and (4) $2a + 2b + c = P$. From (2), $\lambda = \frac{1}{2}c$

and so (1) produces $ca(4a^2 - c^2)^{-1/2} = c \Rightarrow (4a^2 - c^2)^{1/2} = a \Rightarrow 4a^2 - c^2 = a^2 \Rightarrow$ (5) $c = \sqrt{3}\,a$.

Similarly, since $(4a^2 - c^2)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, (3) gives $\frac{a}{4} - \frac{c^2}{4a} + b = \frac{c}{2}$, so from (5), $\frac{a}{4} - \frac{3a}{4} + b = \frac{\sqrt{3}a}{2}$

$\Rightarrow -\frac{a}{2} - \frac{\sqrt{3}a}{2} = -b \Rightarrow$ (6) $b = \frac{a}{2}(1 + \sqrt{3})$. Substituting (5) and (6) into (4) we get:

$2a + a(1 + \sqrt{3}) + \sqrt{3}\,a = P \Rightarrow 3a + 2\sqrt{3}\,a = P \Rightarrow a = \dfrac{P}{3 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{3}P$ and thus

$b = \dfrac{(2\sqrt{3} - 3)(1 + \sqrt{3})}{6}P = \dfrac{3 - \sqrt{3}}{6}P$ and $c = (2 - \sqrt{3})\,P$.

66. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \Rightarrow \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \left(f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}\right)\mathbf{k}$ (by the Chain

Rule). Therefore