▲ TRUE–FALSE QUIZ ▲

1. This is true by Fubini's Theorem.

2. $\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^{1} e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function. Therefore the statement is true.

3. True:
$$\iint_D \sqrt{4-x^2-y^2}\,dA = \text{ the volume under the surface } x^2+y^2+z^2=4 \text{ and above the } xy\text{-plane}$$
$$= \tfrac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \tfrac{1}{2}\cdot\tfrac{4}{3}\pi(2)^3 = \tfrac{16}{3}\pi$$

4. This statement is true because in the given region, $\left(x^2+\sqrt{y}\right)\sin\left(x^2y^2\right) \le (1+2)(1) = 3$, so $\int_1^4\int_0^1 \left(x^2+\sqrt{y}\right)\sin\left(x^2y^2\right)dx\,dy \le \int_1^4\int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

5. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates, $V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

6. True. The moment of inertia about the z-axis of a solid E with constant density k is $I_z = \iiint_E \left(x^2+y^2\right)\rho(x,y,z)\,dV = \iiint_E \left(kr^2\right)r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

◆ EXERCISES ◆

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A \\ &= \Delta A\left[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)\right.\\ &\qquad \left.+f(2,3)+f(3,1)+f(3,2)+f(3,3)\right]\end{aligned}$$
Using the contour lines to estimate the function values, we have
$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m=n=3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A \\ &= \Delta A\left[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5)\right.\\ &\qquad\left.+f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)\right]\\ &\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2}dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$
$= 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1}dy = \int_0^1 (e^y-1)\,dy = \left[e^y-y\right]_0^1 = e-2$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)\,y\right]_{y=0}^{y=x}dx = \int_0^1 x\cos(x^2)\,dx = \tfrac{1}{2}\sin(x^2)\Big]_0^1 = \tfrac{1}{2}\sin 1$

6. $\int_0^1 \int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 \left(xe^{3x} - x^4\right) dx$

$= \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ (integrating by parts in the first term)

$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi \int_0^1 \left[(y\sin x)\,z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi \int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$

$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

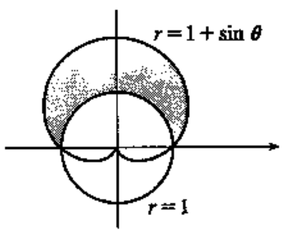
8. $\int_0^1 \int_{\sqrt{y}}^1 \int_0^y xy\,dz\,dx\,dy = \int_0^1 \int_{\sqrt{y}}^1 xy^2\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{x=\sqrt{y}}^{x=1} dy = \int_0^1 \left(\frac{1}{2}y^2 - \frac{1}{2}y^3\right) dy$

$= \left[\frac{1}{6}y^3 - \frac{1}{8}y^4\right]_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y - 4 \le x \le 4 - y, 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

11.



The region whose area is given by $\int_0^\pi \int_1^{1+\sin\theta} r\,dr\,d\theta$ is $\{(r,\theta) \mid 0 \le \theta \le \pi, 1 \le r \le 1 + \sin\theta\}$, which is the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin\theta$.

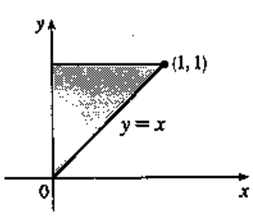
12. The solid is $\left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{6}\right\}$, which lies inside the sphere $\rho = 3$, outside the sphere $\rho = 1$, and within the cone $\phi = \frac{\pi}{6}$.

$$\int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\phi\,d\phi \int_1^3 \rho^2\,d\rho$$

$$= \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/6} \left[\frac{1}{3}\rho^3\right]_1^3$$

$$= (2\pi)\left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{26}{3}\right) = \frac{26\pi}{3}\left(2 - \sqrt{3}\right)$$

13.



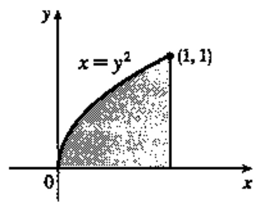
$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx = \int_0^1 \int_0^y e^{x/y}\,dx\,dy$$

$$= \int_0^1 \left[ye^{x/y}\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 (ey - y)\,dy = \left[\frac{e}{2}y^2 - \frac{1}{2}y^2\right]_0^1$$

$$= \frac{1}{2}(e - 1)$$

14.



$$\int_0^1\int_{y^2}^1 y\sin(x^2)\,dx\,dy = \int_0^1\int_0^{\sqrt{x}} y\sin(x^2)\,dy\,dx = \int_0^1 \tfrac{1}{2}x\sin(x^2)\,dx = \left[-\tfrac{1}{4}\cos(x^2)\right]_0^1 = \tfrac{1}{4}(1-\cos 1)$$

15. $$\int_2^4\int_0^1 \frac{1}{(x-y)^2}\,dx\,dy = \int_2^4 \left[-(x-y)^{-1}\right]_{x=0}^{x=1} dy = \int_2^4 \left(-\frac{1}{y} - \frac{1}{1-y}\right) dy = \left[-\ln y + \ln|1-y|\right]_2^4 = -\ln 4 + \ln 3 + \ln 2 = \ln\tfrac{3}{2}$$

16. $\int_{-1}^1\int_{x^2-1}^{x+1} x^3\,dy\,dx = \int_{-1}^1 (2x^3 + x^4 - x^5)\,dx = \left[\frac{1}{2}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6\right]_{-1}^1 = \frac{2}{5}$

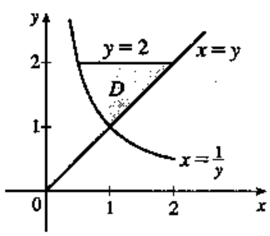
17. The curves $y^2 = x^3$ and $y = x$ intersect when $x^3 = x$, that is when $x = 0$ and $x = 1$ (note that $x \neq -1$ since $x^3 = y^2 \;\Rightarrow\; x \geq 0$.) So $\int_0^1\int_{x^{3/2}}^x xy\,dy\,dx = \int_0^1 \left[\frac{1}{2}x^3 - \frac{1}{2}x^4\right] dx = \left[\frac{1}{8}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{1}{40}$.

18. $\int_0^1\int_0^{x^2} xe^y\,dy\,dx = \int_0^1 x\left(e^{x^2} - 1\right) dx = \frac{1}{2}\left(e^{x^2} - x^2\right)\Big]_0^1 = \dfrac{e-2}{2}$

· x, and

19. $$\int_0^1\int_0^{1-y^2} (xy + 2x + 3y)\,dx\,dy = \int_0^1 \left[\tfrac{1}{2}x^2y + x^2 + 3xy\right]_{x=0}^{x=1-y^2} dy = \int_0^1 \left[\tfrac{1}{2}y(1-y^2)^2 + (1-y^2)^2 + 3y(1-y^2)\right] dy = \tfrac{1}{2}y^5 + y^4 - 4y^3 - 2y^2 + \tfrac{7}{2}y + 1 = \tfrac{1}{5} + \tfrac{1}{12} - \tfrac{2}{3} + \tfrac{7}{4} = \tfrac{41}{30}$$

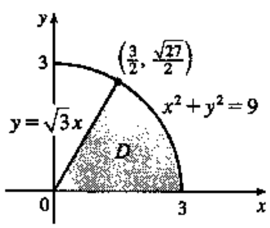
20.



$$\iint_D y\,dA = \int_1^2\int_{1/y}^y y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right) dy = \int_1^2 (y^2 - 1)\,dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2 = \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

the

21.



$$\iint_D (x^2+y^2)^{3/2}\,dA = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2}\,r\,dr\,d\theta = \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = [\theta]_0^{\pi/3}\left[\frac{1}{5}r^5\right]_0^3 = \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$

22. The circle bounding the disk is given by $x^2 + (y-1)^2 = 1$ or $x^2 + y^2 = 2y$ and in polar coordinates $r = 2\sin\theta$. Thus $\iint_D \sqrt{x^2+y^2}\,dA = \int_0^\pi\int_0^{2\sin\theta} r^2\,dr\,d\theta = \int_0^\pi \frac{8}{3}\sin^3\theta\,d\theta = \frac{8}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^\pi = \frac{32}{9}$.

23. $\iiint_E x^2z\,dV = \int_0^2\int_0^{2x}\int_0^x x^2z\,dz\,dy\,dx = \int_0^2\int_0^{2x} \frac{1}{2}x^4\,dy\,dx = \int_0^2 x^5\,dx = \frac{1}{6}\cdot 2^6 = \frac{32}{3}$

24. $$\iiint_T y\,dV = \int_0^1\int_0^{2-2x}\int_0^{2-2x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{2-2x} \left[(2-2x)y - y^2\right] dy\,dx = \int_0^1 \left[\tfrac{1}{2}(2-2x)^3 - \tfrac{1}{3}(2-2x)^3\right] dx = \int_0^1 \tfrac{1}{6}(2-2x)^3\,dx = -\tfrac{1}{48}(2-2x)^4\Big]_0^1 = \tfrac{1}{3}$$

25. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2\left(1-y^2-z^2\right)dz\,dy$ 34.

$= \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(r^2\sin^2\theta\right)\left(1-r^2\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{4}\sin^2 2\theta\left(r^5-r^7\right)dr\,d\theta$

$= \int_0^{2\pi}\frac{1}{8}(1-\cos 4\theta)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1}d\theta = \frac{1}{192}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

35.

26. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)\,z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)\left(1-y^2\right)dy$

$= \int_0^1 \frac{1}{2}\left(2-y-y^2+y^3\right)dy = \frac{11}{24}$

27. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\frac{1}{2}y^3\,dy\,dx = \int_0^{\pi}\int_0^2 \frac{1}{2}r^3\left(\sin^3\theta\right)r\,dr\,d\theta$

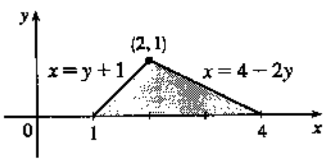
$= \frac{16}{5}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \left(\rho^3\cos^3\phi\right)\rho\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}d\theta\,\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,\int_0^1\rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$ 36.

29. $V = \int_0^2\int_1^4\left(x^2+4y^2\right)dy\,dx = \int_0^2\left[x^2y + \frac{4}{3}y^3\right]_{y=1}^{y=4}dx = \int_0^2\left(3x^2+84\right)dx = 176$

30.



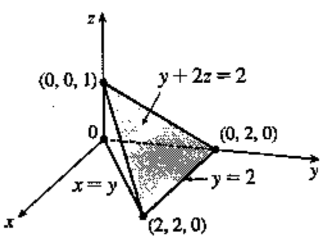
$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1 \frac{1}{3}\left[(4-2y)^3y - (y+1)^3y\right]dy$

$= \int_0^1 3\left(-y^4+5y^3-11y^2+7y\right)dy$

$= 3\left(-\frac{1}{5}+\frac{5}{4}-\frac{11}{3}+\frac{7}{2}\right) = \frac{53}{20}$ 37.

31.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy$

$= \int_0^2\int_0^y\left(1-\frac{1}{2}y\right)dx\,dy$

$= \int_0^2\left(y-\frac{1}{2}y^2\right)dy = \frac{2}{3}$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left(3r - r^2\sin\theta\right)dr\,d\theta = \int_0^{2\pi}\left[6-\frac{8}{3}\sin\theta\right]d\theta$

$= 6\theta\big]_0^{2\pi} + 0 = 12\pi$

33. Using the wedge above the plane $z=0$ and below the plane $z=mx$ and noting that we have the same volume for $m<0$ as for $m>0$ (so use $m>0$), we have 38.

$$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\frac{1}{2}m\left(a^2-9y^2\right)dy = m\left[a^2y-3y^3\right]_0^{a/3}$$

$$= m\left(\frac{1}{3}a^3 - \frac{1}{9}a^3\right) = \frac{2}{9}ma^3$$

34. The paraboloid and the half-cone intersect when $x^2+y^2=\sqrt{x^2+y^2}$, that is when $x^2+y^2=1$ or 0. So

$$V=\iint_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}}dz\,dA=\int_0^{2\pi}\int_0^1\int_{r^2}^{r}r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left(r^2-r^3\right)dr\,d\theta$$
$$=\int_0^{2\pi}\left(\tfrac{1}{3}-\tfrac{1}{4}\right)d\theta=\tfrac{1}{12}(2\pi)=\tfrac{\pi}{6}$$

35. (a) $m=\int_0^1\int_0^{1-y^2}y\,dx\,dy=\int_0^1\left(y-y^3\right)dy=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

(b) $M_y=\int_0^1\int_0^{1-y^2}xy\,dx\,dy=\int_0^1\frac{1}{2}y\left(1-y^2\right)^2dy=-\frac{1}{12}\left(1-y^2\right)^3\Big]_0^1=\frac{1}{12}$,
$M_x=\int_0^1\int_0^{1-y^2}y^2\,dx\,dy=\int_0^1\left(y^2-y^4\right)dy=\frac{2}{15}$. Hence $(\overline{x},\overline{y})=\left(\frac{1}{3},\frac{8}{15}\right)$.

(c) $I_x=\int_0^1\int_0^{1-y^2}y^3\,dx\,dy=\int_0^1\left(y^3-y^5\right)dy=\frac{1}{12}$,
$I_y=\int_0^1\int_0^{1-y^2}yx^2\,dx\,dy=\int_0^1\frac{1}{3}y\left(1-y^2\right)^3dy=-\frac{1}{24}\left(1-y^2\right)^4\Big]_0^1=\frac{1}{24}$.

36. (a) $m=\frac{1}{4}\pi Ka^2$ where K is constant,
$M_y=\iint_{x^2+y^2\le a^2}Kx\,dA=K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta=\frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta=\frac{1}{3}a^3K$, and
$M_x=K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta=\frac{1}{3}a^3K$ (by symmetry $M_y=M_x$). Hence the centroid is
$(\overline{x},\overline{y})=\left(\frac{4}{3\pi}a,\frac{4}{3\pi}a\right)$.

(b) $m=\int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta=\left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right)=\frac{1}{15}a^5$,
$M_y=\int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta=\frac{1}{8}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right)=\frac{1}{96}\pi a^6$, and
$M_x=\int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta=\left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right)=\frac{1}{24}a^6$. Hence $(\overline{x},\overline{y})=\left(\frac{5}{32}\pi a,\frac{5}{8}a\right)$.

37. (a) The equation of the cone with the suggested orientation is $(h-z)=\frac{h}{a}\sqrt{x^2+y^2}$, $0\le z\le h$. Then $V=\frac{1}{3}\pi a^2h$ is the volume of one frustum of a cone; by symmetry $M_{yz}=M_{xz}=0$; and

$$M_{xy}=\iint_{x^2+y^2\le a^2}\int_0^{h-(h/a)\sqrt{x^2+y^2}}z\,dz\,dA=\int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)}rz\,dz\,dr\,d\theta$$
$$=\pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr=\frac{\pi h^2}{a^2}\int_0^a\left(a^2r-2ar^2+r^3\right)dr=\frac{\pi h^2}{a^2}\left(\frac{a^4}{2}-\frac{2a^4}{3}+\frac{a^4}{4}\right)=\frac{\pi h^2a^2}{12}$$

Hence the centroid is $(\overline{x},\overline{y},\overline{z})=\left(0,0,\frac{1}{4}h\right)$.

(b) $$I_z=\int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)}r^3\,dz\,dr\,d\theta=2\pi\int_0^a\frac{h}{a}\left(ar^3-r^4\right)dr=\frac{2\pi h}{a}\left(\frac{a^5}{4}-\frac{a^5}{5}\right)=\frac{\pi a^4h}{10}$$

ume for

38. (a) By Definition 12.6.4, the area of S is given by
$A(S)=\int_0^3\int_{-3}^3\sqrt{\left(2u^2\right)^2+(4uv)^2+\left(2v^2\right)^2}\,dv\,du=2\int_0^3\int_{-3}^3\sqrt{u^4+4u^2v^2+v^4}\,dv\,du$.

(b) Using a CAS, we have $2\int_0^3\int_{-3}^3\sqrt{u^4+4u^2v^2+v^4}\,dv\,du\approx 247.8$. (Ask your CAS to evaluate the integral numerically rather than symbolically.)

39. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2 - 2x$, or equivalently $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. We want to find the surface area of the part of the graph of $z = x^2 + y$ that lies over D, so using Equation 12.6.6 we have

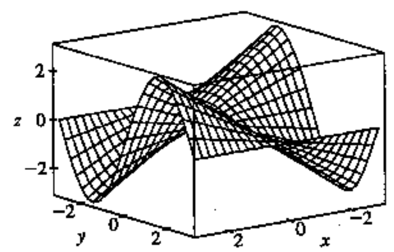
$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2}\, dA \\ &= \int_0^1 \int_0^{2-2x} \sqrt{2 + 4x^2}\, dy\, dx = \int_0^1 \sqrt{2 + 4x^2}\, [y]_0^{2-2x}\, dx = \int_0^1 (2 - 2x)\sqrt{2 + 4x^2}\, dx \\ &= \int_0^1 2\sqrt{2 + 4x^2}\, dx - \int_0^1 2x\sqrt{2 + 4x^2}\, dx \end{aligned}$$

Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\, dx$, we have $\int 2\sqrt{2 + 4x^2}\, dx = x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right)$. If we substitute $u = 2 + 4x^2$ in the second integral, then $du = 8x\, dx$ and $\int 2x\sqrt{2 + 4x^2}\, dx = \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} = \frac{1}{6}\left(2 + 4x^2\right)^{3/2}$. Thus

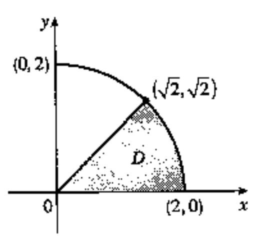
$$\begin{aligned} A(S) &= \left[x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right) - \tfrac{1}{6}\left(2 + 4x^2\right)^{3/2}\right]_0^1 \\ &= \sqrt{6} + \ln\left(2 + \sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} \\ &= \ln\frac{2 + \sqrt{6}}{\sqrt{2}} + \frac{\sqrt{2}}{3} = \ln\left(\sqrt{2} + \sqrt{3}\right) + \frac{\sqrt{2}}{3} \approx 1.6176 \end{aligned}$$

40. Using Formula 12.6.6 with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get

$$\begin{aligned} S &= \int_{-\pi}^{\pi} \int_{-3}^{3} \sqrt{\sin^2 y + x^2 \cos^2 y + 1}\, dx\, dy \\ &\approx 62.9714 \end{aligned}$$



41.



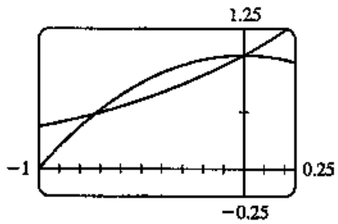
$$\begin{aligned} \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1 + x^2 + y^2}\, dx\, dy &= \int_0^{\pi/4} \int_0^2 \frac{1}{1 + r^2}\, r\, dr\, d\theta \\ &= \int_0^{\pi/4} d\theta \int_0^2 \frac{r}{1 + r^2}\, dr \\ &= [\theta]_0^{\pi/4} \left[\tfrac{1}{2}\ln\left|1 + r^2\right|\right]_0^2 \\ &= \tfrac{\pi}{4}\left(\tfrac{1}{2}\ln 5\right) = \tfrac{\pi}{8}\ln 5 \end{aligned}$$

42. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \left(x^2 + y^2 + z^2\right)^2 dz\, dy\, dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left(\rho^2\right)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

$$= \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{\pi/2} d\theta \int_0^1 \rho^6\, d\rho = [-\cos\phi]_0^{\pi/2}\, [\theta]_0^{\pi/2}\, \left[\tfrac{1}{7}\rho^7\right]_0^1 = 1 \cdot \tfrac{\pi}{2} \cdot \tfrac{1}{7} = \tfrac{\pi}{14}$$

[redacted]

43. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is



$$\iint_D y^2 dA \approx \int_{-0.71}^{0} \int_{e^x}^{1-x^2} y^2\, dy\, dx$$

$$= \tfrac{1}{3} \int_{-0.71}^{0} \left[\left(1 - x^2\right)^3 - e^{3x} \right] dx$$

$$= \tfrac{1}{3} \left[x - x^3 + \tfrac{3}{5} x^5 - \tfrac{1}{7} x^7 - \tfrac{1}{3} e^{3x} \right]_{-0.71}^{0} \approx 0.0512$$

44. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2} y + \frac{1}{3} z = 1$, or $z = 3 - 3x - \frac{3}{2} y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is $m = \iiint_T \rho(x, y, z)\, dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} \left(x^2 + y^2 + z^2\right) dz\, dy\, dx = \frac{7}{5}$. The center of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(m^{-1} \iiint_T x\rho(x, y, z)\, dV, m^{-1} \iiint_T y\rho(x, y, z)\, dV, m^{-1} \iiint_T z\rho(x, y, z)\, dV\right)$$

$$= \left(\tfrac{4}{21}, \tfrac{11}{21}, \tfrac{8}{7}\right)$$

45. (a) $f(x, y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x, y)\, dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 3] \times [0, 2]$, we can say

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_0^3 \int_0^2 C(x + y)\, dy\, dx$$

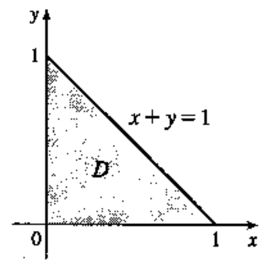
$$= C \int_0^3 \left[xy + \tfrac{1}{2} y^2 \right]_{y=0}^{y=2} dx = C \int_0^3 (2x + 2)\, dx = C\left[x^2 + 2x\right]_0^3 = 15C$$

Then $15C = 1 \;\Rightarrow\; C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2} \int_{1}^{\infty} f(x, y)\, dy\, dx = \int_0^2 \int_1^2 \frac{1}{15}(x, y)\, dy\, dx = \frac{1}{15} \int_0^2 \left[xy + \frac{1}{2} y^2 \right]_{y=1}^{y=2} dx$

$$= \tfrac{1}{15} \int_0^2 \left(x + \tfrac{3}{2}\right) dx = \tfrac{1}{15} \left[\tfrac{1}{2} x^2 + \tfrac{3}{2} x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X + Y \le 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X + Y \le 1) = \iint_D f(x, y)\, dA = \int_0^1 \int_0^{1-x} \tfrac{1}{15}(x + y)\, dy\, dx$$

$$= \tfrac{1}{15} \int_0^1 \left[xy + \tfrac{1}{2} y^2 \right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{15} \int_0^1 \left[x(1 - x) + \tfrac{1}{2}(1 - x)^2 \right] dx$$

$$= \tfrac{1}{30} \int_0^1 \left(1 - x^2\right) dx = \tfrac{1}{30} \left[x - \tfrac{1}{3} x^3 \right]_0^1 = \tfrac{1}{45}$$

46. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800} e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

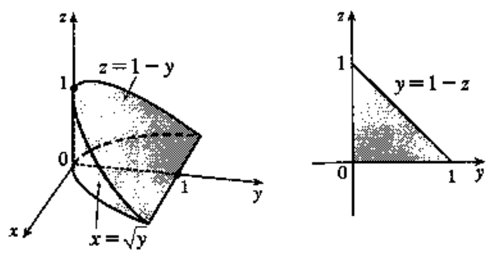
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

$$f(x, y, z) = \begin{cases} \frac{1}{800^3} e^{-(x+y+z)/800} & \text{if } x \ge 0, y \ge 0, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X+Y+Z \leq 1000)$, or equivalently $P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1000$. The plane $x+y+z=1000$ meets the xy-plane in the line $x+y=1000$, so we have

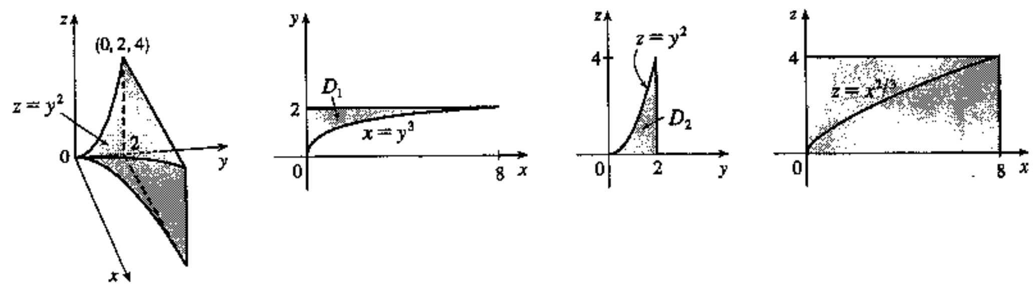
$$\begin{aligned}
P(X+Y+Z \leq 1000) &= \iiint_E f(x,y,z)\,dV \\
&= \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \tfrac{1}{800^3} e^{-(x+y+z)/800}\,dz\,dy\,dx \\
&= \tfrac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y} dy\,dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\int_0^{1000-x}\left[e^{-5/4} - e^{-(x+y)/800}\right] dy\,dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}y + 800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x} dx \\
&= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}(1800-x) - 800e^{-x/800}\right] dx \\
&= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(1800-x)^2 + 800^2e^{-x/800}\right]_0^{1000} \\
&= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(800)^2 + 800^2e^{-5/4} + \tfrac{1}{2}e^{-5/4}(1800)^2 - 800^2\right] \\
&= 1 - \tfrac{97}{32}e^{-5/4} \approx 0.1315
\end{aligned}$$

47.



$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$

48.



$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where
$E = \{(x,y,z) \mid 0 \leq y \leq 2, 0 \leq x \leq y^3, 0 \leq z \leq y^2\}$. If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \{(x,y) \mid 0 \leq y \leq 2, 0 \leq x \leq y^3\} = \{(x,y) \mid 0 \leq x \leq 8, \sqrt[3]{x} \leq y \leq 2\}$,
$D_2 = \{(y,z) \mid 0 \leq z \leq 4, \sqrt{z} \leq y \leq 2\} = \{(y,z) \mid 0 \leq y \leq 2, 0 \leq z \leq y^2\}$,
$D_3 = \{(x,z) \mid 0 \leq x \leq 8, 0 \leq z \leq 4\}$.