Math 21a Practice Final Exam

(Fall 2000)

1) __ 2) __ 3) __ 4) __ 5) __ 6) __ 7) __ 8) __ 9) __ 10) __ 11) __ 12) __ : Total ______

Name: ____

Section TF:

Allcock • Chen • Karigiannis • Knill • Liu • Rasmussen • Rogers • Taubes • Winter • Winters

Instructions:

- Print your name in the line above and circle the name of your section TF.

- **THERE ARE THREE PARTS TO THIS EXAM:**

A. All students answer Part A questions.

B. Only students in either the Regular or Physics sections answer Part B questions.

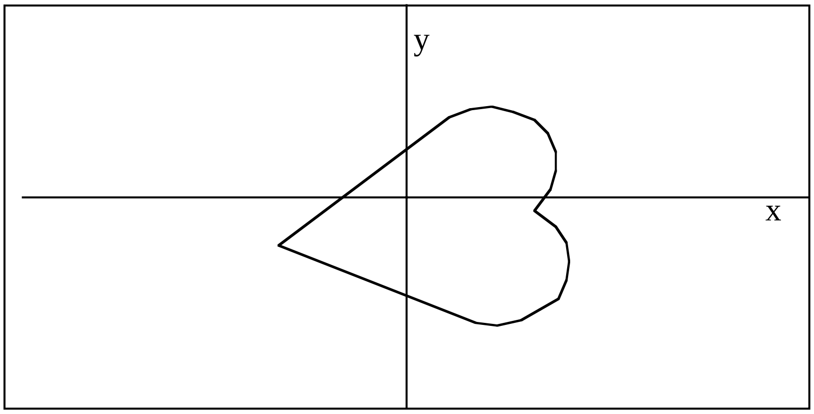
C. Only students in the BioChem sections answer Part C questions.

- Answers are to be written on the same page as the question. However, if more space is needed, use the back of the facing page. Use pages with unanswered questions (either those in Part C or in Part B, as the case may be) if additional space is needed.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.
- All problems will count the same amount.
- Neither unstaple nor remove pages from your exam booklet.

In agreeing to take this exam, you are implicitly accepting Harvard University's honor code.

PART A: The questions in this part of the exam should be answered by all students.

1. Pictured below is a curve in the x-y plane on which a flea lives. The temperature of the plane on a region containing the curve is T(x, y) = 20 + x + y.



a) Mark all of the points on the curve where the curve's temperature is clearly a local maximum.
b) Mark all of the points on the curve where the curve's temperature is clearly a local minimum.
c) If the flea travels counter clockwise around the curve at constant speed, label all of the points on the curve where the temperature is increasing at the fastest rate.
d) For the same flea as above, label all of the points on the curve where the temperature is decreasing at the fastest rate.
e) Mark all of the points on the curve which could plausibly have zero rate of change of the temperature without being a local maximum or minimum.

2. First use Cartesian coordinates, then use cylindrical coordinates, and finally use spherical coordinates to write iterated triple integral expressions for the integral of xyz over the region in space where $0 \leq z \leq (1 - x^2 - y^2)^{1/2}$, $x^2 + y^2 \leq 1$ and $x - y \geq 0$ and $y \geq 0$. Do not evaluate your expressions.

3. Suppose that $f(x, y, z)$ is a function on R^3 whose value at the origin is 6. Suppose that at the origin, the dot product of ∇f with (1, 3, 1) equals 1, the dot product of ∇f with (2, 1, 3) equals 2, and with (1, 2, 3) equals 3. Write down:
   a) A non-zero vector which is tangent at the origin to the level set $f = 6$.
   b) The equation for the tangent plane at the origin to the level set $f = 6$.
   c) The linear approximation to f at the origin.
   d) The directional derivative at the origin of f in the direction $(1, 1, 1)/3^{1/2}$.

4. Integrate the function xyz over region where $x \geq 0$, $y \geq 0$, $z \geq 0$ and $z + x^2 + y^2 \leq 1$.

5. Find all points in the region $x^2 + y^2 + z^2 \leq 1$ at which the gradient of $x^2/16 + y^2/4 + z^2$ has largest length (that is, largest magnitude).

6. Give a parametric equation for the line which lies in the planes $x + y - z = 1$ and $x + 2y + z = 1$.

7. Integrate the function $\cos(\theta)$ over region of the plane which is given, in polar coordinates, by the condition that $\cos^2(\theta) \leq r \leq \cos(\theta)$.

8. Suppose that a function u obeys the equation

$$u_{xx} + u_{yy} = 1$$

   where $x^2 + y^2 \leq 1$. Note that there are infinitely many functions which obey this equation. Here are two examples: $u = (x^2 + y^2)/4$ and $u = x^2/6 + y^2/3 - x/27$.

   Label each of the following statements with 'A' if the statement is true for every possible solution, 'S' if it is true for some but not all solutions, or 'N' if the statement is not true for any solution. Justify your answer in each case.
   a) The gradient of u must be non-zero at some point.
   b) The function u is independent of x.
   c) The function u achieves its global maximum where $x^2 + y^2 = 1$.
   d) The function u obeys $u(0) > 0$ and $u < 0$ where $x^2 + y^2 > 1/2$.
   e) The function u achieves its global minimum where $x^2 + y^2 = 1$.

PART B: The questions in this part of the exam are only for students in either the Regular or the Physics sections. Students in the BioChem sections do not answer questions in this part of the exam; they turn immediately to Part C of the exam.

9. Let **F** = $(x + xz, y - yz, z^2)$. Here are two surfaces in R^3 with the same boundary:

   A: $z = (1 - x^2 - y^2)^{1/2}$ with $x^2 + y^2 \leq 1$.
   B: $z = 2(1 - x^2 - y^2)^{1/2}$ with $x^2 + y^2 \leq 1$.

   a) Label the surfaces with 1 or 2; where 1 has the lowest flux of **F** and 2 the highest. Ties are allowed. Use the normal which has positive dot product with (0, 0, 1). Justify your answer.
   b) Label the surfaces with 1 or 2; where 1 has the lowest flux of curl(**F**) and 2 has the highest. Ties are allowed here too. Use the same normal as in Part a. Justify your answer.
   c) Compute the flux of curl(**F**) through the surface where $z = 1 - x^2 - y^2$ with $x^2 + y^2 \leq 1$.

10. a) Write down a 2-variable, iterated integral for the area of the portion of the surface where $x^2 - 4(y^2 + z^2) = 0$ and $0 \leq x \leq 1$. Don't evaluate your integral.
    b) Write down a 2-variable, iterated integral for the flux of **F** = (x, 0, 0) through the surface as defined using the normal vector whose dot product with (1, 0, 0) is negative.

11. In each case below, either write down a vector field on R^3 with the desired properties, or else explain why no such vector field exists.
    a) The curl is (1, 1, 0) and the flux is zero through any surface in a plane where z is constant.
    b) The curl is (1, 1, 0) and the line integral is zero around all loops in the x = 0 plane.
    c) The divergence is 2 and the line integral is zero around all closed loops.
    d) The divergence is 2 and the flux through any closed surface is zero.

12. a) Write down a vector field on R^2 whose line integral around the counter-clockwise oriented boundary of every bounded region in the plane gives the integral of x^2 over that region.
    b) Parametrize the circle where $x^2 + y^2 = 1$ and directly compute the line integral of your vector field around this circle.

PART C: The questions in this part of the exam are only for students in the BioChem sections. Students in either the Regular or Physics sections do not answer questions in this part of the exam.

9. Guildenstern flips a biased coin that lands heads 70% of the time. He makes 80 flips.
   a) What are the expected number of heads and the standard deviation?
   b) Give an exact expression for the probability that there are 62 or more heads in this experiment. Don't compute the numerical value of your expression.
   c) Write an integral using a normal distribution whose value gives an approximate answer for the probability in Part b. Don't evaluate the integral.

10. Consider an experiment which consists of throwing three dice, each with sides numbered 1-6.
    a) What is the probability of getting more than 2 on at least one of the dice?
    b) What is the probability that the sum showing on all three dice is less than or equal to 4?
    c) Given that the first die shows 3, what is the probability that the sum of all three dice is 7?

11. In the state of Massachusetts, the mortality rate due to a certain rare cancer is 25 persons in a four year period (as measured over a long period of years).
    a) Use a Poisson distribution to give an expression for the probability of k deaths due to this cancer in a two year period. Don't compute the numerical value of your expression.
    b) If 40 deaths are reported in a given four year period, how many standard deviations is this from the mean of the Poisson distribution that you used to answer Part a.
    c) Using your Poisson distribution, give an expression (but don't evaluate it) for the probability of seeing 40 deaths in both the 4-year period that started in 1992 and in the 4-year period that started in 1996.

12. A group of people were tested to study the relationship between heart disease and physical fitness. Here is the data showing the relationship between exercise heart rate and heart disease:

Heart disease (per 100 people)	Heart rate
9	≤ 105
8	106-115
12	116-126
13	≥ 127

Suppose that the percentages of the overall population in each category are 20%, 30%, 40% and 10%, respectively. Let the positive test for heart disease be a heart rate of 127 or more and a negative test a rate of 126 or less.

a) What is the overall rate of heart disease in the population?
b) Compute the sensitivity and the predictive-value-positive (PV^+) of the test.
c) Given a negative test, what is the probability that a person has heart disease?