Answers to 2la Exam #1 Wed Mar 7 2001

(1) Given the following points: P(1, 2,-2), Q(-3, 1, -14), R(-1, 2,-6), and S(1,-2,-8). Are these points coplanar, i.e. do they all lie in a common plane in $\mathbf{R}^3$? Justify your answer.

There are several ways to go about this – one way is to calculate the equation of a plane through 3 points and see if the fourth point is on the same plane, i.e. plane through P, Q and R – Find vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$\langle -3-1, 1-2, -14-(-2)\rangle = \langle -4, -1, -12\rangle$ and

$\langle -1-1, 2-2, -6-(-2)\rangle = \langle -2, 0, -4\rangle$,

so a normal vector to the plane is given by the cross product of these two vectors:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -1 & -12 \\ -2 & 0 & -4 \end{vmatrix} = 4\vec{i} + 8\vec{j} - 2\vec{k} = \langle 4, 8, -2\rangle$$

So the plane through P, Q and R goes through $P = (1, 2, -2)$ with normal $\langle 4, 8, -2\rangle$,

so has equation $4(x-1) + 8(y-2) - 2(z+2) = 0$

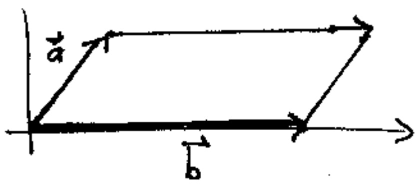
Now check whether $S = (1, -2, -8)$ satisfies this equation to see if S is on this plane also,

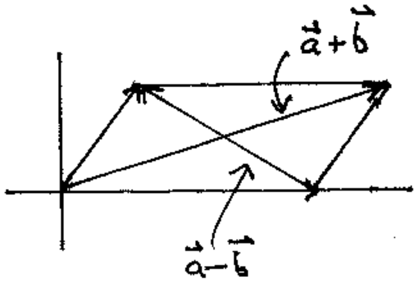
$4(1-1) + 8(-2-2) - 2(-8+2) = 0 - 32 + 12 \neq 0$,

so no, the points are not coplanar.

(2) Use vectors to show that the diagonals of a parallelogram are perpendicular if and only if all four sides of the parallelogram have equal length. (*Such a figure is called a rhombus.*)

Create the parallelogram with two vectors $\vec{a}$ and $\vec{b}$ as follows:





so the question is when are vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ perpendicular?

check with dot products:

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$$

implies

$$\vec{a}\cdot\vec{a}+\cancel{\vec{b}\cdot\vec{a}}-\cancel{\vec{a}\cdot\vec{b}}-\vec{b}\cdot\vec{b}=0$$

so $\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b}$ if
$\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular

but this means

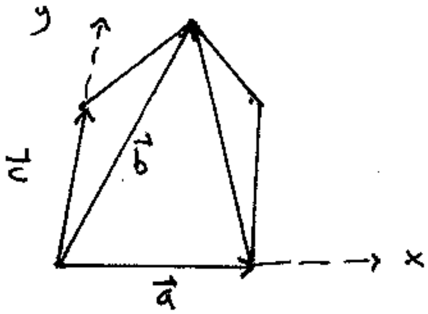
$$|\vec{a}|^2 = |\vec{b}|^2 \quad \text{or}$$

$$|\vec{a}| = |\vec{b}| \quad \left(\text{since both sides are positive}\right)$$

$\Rightarrow$ sides have equal length.

(3) Consider a pyramid-shaped tent of height 5 meters and with a 4 meter by 4 meter square base

(a) The top of the tent lies directly over the center of the base. Find the angle between two adjacent triangular faces of the tent, i.e. the angle between their outward pointing normal vectors



Thus if $\vec{a} = \langle 4,0,0 \rangle$, $\vec{c} = \langle 0,4,0 \rangle$ then $\vec{b}$ vector pointing to top of tent $= \langle 2,2,5 \rangle$, and we need to find the angle between $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{c}$ (make sure you end up with outward facing normals!)

$$\vec{a}\times\vec{b} = \langle 0,-20,8 \rangle \quad \text{and} \quad \vec{b}\times\vec{c} = \langle -20,0,8 \rangle$$

Then $(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c}) = 64 = |\vec{a}\times\vec{b}|\,|\vec{b}\times\vec{c}|\cos(\theta)$

θ = angle between

so $\cos(\theta) = \dfrac{64}{\sqrt{464}\cdot\sqrt{464}} = \dfrac{64}{464} = \dfrac{4}{29}$

and $\theta = \cos^{-1}\left(\frac{4}{29}\right)$ ($\approx 82°$, a little less than $90°$ as we'd expect, although without calculators you probably wouldn't notice this! ... well $\cos^{-1}(0) = 90°$ so $\cos^{-1}\left(\frac{4}{29}\right) < 90°$)

(b) If the tent is made of canvas and consists of the four triangular sides plus the square bottom, find the total area of canvas used to make the tent.

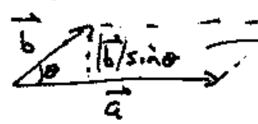
Well the bottom's pretty easy!
$= 5^2 = 25$ sq. meters

4 sides, and we know each side has

area $= \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\sqrt{464}$, so in total

area $= 25 + 2\sqrt{464}$ sq. meters

→ parallelogram interpretation of cross product



Area $= |\vec{a}|\,(|\vec{b}|\sin\theta) = |\vec{a}\times\vec{b}|$

(4) Consider the vector field $\mathbf{F}(x, y) = (y + 2x)\mathbf{i} + (x - 3)\mathbf{j}$. We wish to find the work integral $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ along the path γ, where γ is the line segment from the point (0, 2) to the point (4,-2).

(a) Sketch the direction of $\mathbf{F}$ at the five points (0, 2), (1, 1), (2, 0), (3,-1), and (4,-2). On the basis of this sketch, explain whether you expect the value of the integral to be positive, negative, or zero.

Not necessary for first midterm
Spring 2002

(b) Evaluate the integral $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ along the line segment from the point (0, 2) to (4,-2).

— ditto —

(5) Consider the function $f(x, y) = \sqrt{50 - x^2 - y^2}$.

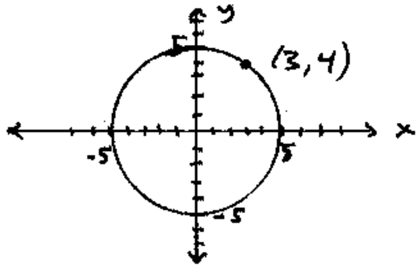
(a) Describe geometrically the surface formed by the graph $z = f(x, y)$.

Since $z = \sqrt{50 - x^2 - y^2}$ then $z^2 = 50 - x^2 - y^2$ (with $z \geq 0$) so $x^2 + y^2 + z^2 = 50$, and this gives the upper half of a sphere of radius $\sqrt{50}$

(b) Sketch and label the contour of the function f through the point $(x, y) = (3, 4)$.

Level curves (contours) are just circles (since horizontal slices through the sphere yield circles

so ;



(c) Calculate ∇f , the gradient of f, at the point $(x, y) = (3, 4)$ and indicate it on your diagram above.

not necessary for spring 2002 midterm / practise

(d) Calculate the directional derivative of f at the point (3, 4) in the direction of the vector (2,-1).

— Ditto —

(e) Find an equation for the plane tangent to the graph $z = f(x, y)$ at the point <u>on the graph</u> where $x = 3$ and $y = 4$.

find normal vector → note it's just the vector from the origin to the point (3,4,5) on the sphere

so tangent plane goes through (3,4,5) normal to $\langle 3,4,5 \rangle$ equation is just

$$3(x-3) + 4(y-4) + 5(z-5) = 0$$

(6) The Texahoma Saloon is situated on the Texas-Oklahoma border. Its roof is perpendicular to the vector (1,-2, 2) and includes the point with coordinates (-2, 4, 5). [The x-axis runs from west to east, the y-axis from south to north, and the z-axis points vertically upward.] Marshall Wyatt Zorn has been sent from Minnesota to this border country to enforce a recent executive order decreeing that all distances are to be measured in meters

(a) Write an equation satisfied by the coordinates (x, y, z) of any point on the roof.

$$1\left(x-(-2)\right) + (-2)(y-4) + 2(z-5) = 0$$

$$\left(\text{i.e. } \left(\langle x,y,z\rangle - \langle -2,4,5\rangle\right)\cdot\langle 1,-2,2\rangle = 0\right)$$

(b) The state line lies along the plane $y = 4$ (with $y > 4$ in Oklahoma and $y < 4$ in Texas, of course). Marshall Zorn is asked to paint the state line on the saloon's roof. Find parametric equations for this line. (You do not have to specify a parameter range.)

When $y=4$ in the equation of the plane from part (a) we are left with $1(x+2)-2(4-4)+2(z-5)=0$ or $x+2+2z-10=0$, so $2z=-x+8$, $z=-\frac{1}{2}x+4$ and so with x as the parameter the line is given as $\langle x, 4, -\frac{1}{2}x+4\rangle$, or $x=t,\ y=4,\ z=-\frac{1}{2}t+4$ parametric equations

(c) Prominently posted in the saloon is a sign that states "No firearms within 2 meters of the roof." Calamity W Jane, while trying to make a point about the Second Amendment, holds her Colt 45 at the point (-3,-3, 2) Marshall Zorn needs to decide whether she is in violation of the law. Help him out by calculating the distance from the Colt to the roof.

If $P=(-2,4,5)$ the point given in part (a) on the roof, then if $Q=(-3,-3,2)$, where the gun is, then distance from Q to plane is given as the scalar projection of $\overrightarrow{PQ}$ onto the normal vector $\langle 1,-2,2\rangle$ (absolute value of the result),

i.e. $\text{distance} = \left|\text{comp}_{\vec{n}}\left(\overrightarrow{PQ}\right)\right| = \dfrac{|\langle 1,-2,2\rangle\cdot\langle -1,-7,-3\rangle|}{|\langle 1,-2,2\rangle|} = \dfrac{7}{3} > 2$ meters

(d) In panic at the prospect of being arrested, Jane fires her Colt 45. The bullet travels with great velocity proportional to the vector (5, 6, 2) and moves in a straight line through the roof. Determine the location of the hole that it makes.

parametrize line of bullet's travel: $\langle -3,-3,2\rangle + t\langle 5,6,2\rangle$ or $x=-3+5t$ $y=-3+6t$ $z=2+2t$, when bullet hits roof these three components satisfy the roof's equation given in part (a): $x+2-2(y-4)+2(z-5)=0$ or $x-2y+2z=0$ so $(-3+5t)-2(-3+6t)+2(2+2t)=0$ or $-3t+7=0$, so $t=\frac{7}{3}$ so gun's bullet hits roof at $\left(-3+\frac{35}{3}, -3+\frac{42}{3}, 2+\frac{14}{3}\right) = \left(\frac{26}{3}, 11, \frac{20}{3}\right)$