◆13◆ Review

1. See Definitions 1 and 2 in Section 13.1. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field F is a vector field which is the gradient of some scalar function f.

   (b) The function f in part (a) is called a potential function for F, that is, $F = \nabla f$.

3. (a) See Definition 13.2.2.

   (b) We normally evaluate the line integral using Formula 13.2.3.

   (c) The mass is $m = \int_C \rho(x, y) \, ds$, and the center of mass is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{m} \int_C x\rho(x, y) \, ds$, $\bar{y} = \frac{1}{m} \int_C y\rho(x, y) \, ds$.

   (d) See (5) and (6) in Section 13.2 for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) on page 930).

   (e) For plane curves, see Equations 13.2.7. We have similar results for space curves (see the equation preceding (10) on page 930).

4. (a) See Definition 13.2.13.

   (b) If F is a force field, $\int_C F \cdot dr$ represents the work done by F in moving a particle along the curve C.

   (c) $\int_C F \cdot dr = \int_C P \, dx + Q \, dy + R \, dz$

5. See Theorem 13.3.2.

6. (a) $\int_C F \cdot dr$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

   (b) See Theorem 13.3.4.

7. See the statement of Green's Theorem on page 945.

8. See Equations 13.4.5.

9. (a) $\operatorname{curl} F = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right) i + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right) j + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) k = \nabla \times F$

   (b) $\operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \nabla \cdot F$

   (c) For curl F, see the discussion accompanying Figure 1 on page 955 as well as Figure 6 and the accompanying discussion on page 975. For div F, see the discussion following Example 5 on page 956 as well as the discussion preceding (8) on page 982.

10. See Theorem 13.3.6; see Theorem 13.5.4.

11. (a) See (1) in Section 13.6.

   (b) We normally evaluate the surface integral using Formula 13.6.2.

   (c) See Formula 13.6.4.

   (d) The mass is $m = \iint_S \rho(x, y, z) \, dS$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z) \, dS$, $\bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z) \, dS$, $\bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z) \, dS$.

12. (a) See Figures 6 and 7 and the accompanying discussion in Section 13.6. A Möbius strip is a nonorientable surface; see Figures 4 and 5 and the accompanying discussion on page 964.

(b) See Definition 13.6.8.

(c) See Formula 13.6.9.

(d) See Formula 13.6.10.

13. See the statement of Stokes' Theorem on page 971.

14. See the statement of the Divergence Theorem on page 978.

15. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

--------------------------------------- ▲ **TRUE–FALSE QUIZ** ▲ ---------------------------------------

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 13.5.1.)

3. True, by Theorem 13.5.3 and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 13.3.2.

5. False. See Exercise 13.3.33. (But the assertion is true if D is simply-connected; see Theorem 13.3.6.)

6. False. See the discussion accompanying Figure 8 on page 929.

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

8. False by Theorem 13.5.11, because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

--------------------------------------- ◆ **EXERCISES** ◆ ---------------------------------------

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}(P)$ is positive.

2. We can parametrize C by $x = x,\ y = x^2,\ 0 \le x \le 1$ so
$$\int_C x\, ds = \int_0^1 x \sqrt{1 + (2x)^2}\, dx = \tfrac{1}{12}\left(1 + 4x^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(5\sqrt{5} - 1\right).$$

3. $\int_C x^3 z\, ds = \int_0^{\pi/2} (2\sin t)^3\, (2\cos t) \sqrt{(2\cos t)^2 + (1)^2 + (-2\sin t)^2}\, dt = \int_0^{\pi/2} \left(16\sin^3 t \cos t\right)\sqrt{5}\, dt$
$$= 4\sqrt{5}\sin^4 t\Big]_0^{\pi/2} = 4\sqrt{5}$$

4. $\int_C xy\, dx + y\, dy = \int_0^{\pi/2} (x\sin x + \sin x \cos x)\, dx = -x\cos x + \sin x - \tfrac{1}{4}\cos 2x\Big]_0^{\pi/2} = \tfrac{3}{2}$

5. $x = \cos t \Rightarrow dx = -\sin t\, dt,\ y = \sin t \Rightarrow dy = \cos t\, dt,\ 0 \le t \le 2\pi$ and
$\int_C x^3 y\, dx - x\, dy = \int_0^{2\pi} \left(-\cos^3 t \sin^2 t - \cos^2 t\right) dt = \int_0^{2\pi} \left(-\cos^3 t \sin^2 t - \cos^2 t\right) dt = -\pi$
Or: Since C is a simple closed curve, apply Green's Theorem giving
$\iint_{x^2 + y^2 \le 1} \left(-1 - x^3\right) dA = \int_0^1 \int_0^{2\pi} \left(-r - r^4 \cos^3 \theta\right) d\theta = -\pi.$

6. $\int_C x \sin y \, dx + xyz \, dz = \int_0^1 \left(t \sin t^2 + 3t^8 \right) dt = -\frac{1}{2} \cos t^2 + \frac{1}{3} t^9 \Big]_0^1 = \frac{5}{6} - \frac{1}{2} \cos 1$

7.

C_1: $x = t$, $y = t$, $z = 2t$, $0 \leq t \leq 1$;

C_2: $x = 1 + 2t$, $y = 1$, $z = 2 + 2t$, $0 \leq t \leq 1$.
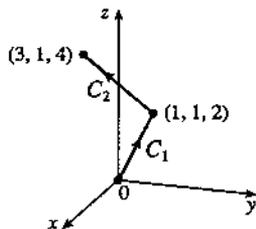
Then $\int_C y \, dx + z \, dy + x \, dz = \int_0^1 5t \, dt + \int_0^1 (4 + 4t) \, dt = \frac{17}{2}$.

8. $\mathbf{F}(\mathbf{r}(t)) = -t^7 \, \mathbf{i} + e^{-t^3} \, \mathbf{j}$, $\mathbf{F} \cdot \mathbf{r}'(t) = -2t^8 - 3t^2 e^{-t^3}$ and

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(-2t^8 - 3t^2 e^{-t^3} \right) dt = -\frac{2}{9} t^9 + e^{-t^3} \Big]_0^1 = e^{-1} - \frac{11}{9}$.

9. $\mathbf{F}(\mathbf{r}(t)) = \left(2t + t^2 \right) \mathbf{i} + t^4 \, \mathbf{j} + 4t^4 \, \mathbf{k}$, $\mathbf{F} \cdot \mathbf{r}'(t) = 4t + 2t^2 + 2t^5 + 16t^7$ and

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(4t + 2t^2 + 2t^5 + 16t^7 \right) dt = 5$.

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2} t$, $z = 3t$, $0 \leq t \leq 1$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[3t \, \mathbf{i} + (3 - 3t) \mathbf{j} + \frac{\pi}{2} t \, \mathbf{k} \right] \cdot \left[-3\mathbf{i} + \frac{\pi}{2} \mathbf{j} + 3\mathbf{k} \right] dt = \int_0^1 \left[-9t + \frac{3\pi}{2} \right] dt$$

$$= \frac{1}{2}(3\pi - 9)$$

(b) $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (3 \sin t \, \mathbf{i} + 3 \cos t \, \mathbf{j} + t \, \mathbf{k}) \cdot (-3 \sin t \, \mathbf{i} + \mathbf{j} + 3 \cos t \, \mathbf{k}) \, dt$

$$= \int_0^{\pi/2} \left(-9 \sin^2 t + 3 \cos t + 3t \cos t \right) dt$$

$$= \left[-\frac{9}{2}(t - \sin t \cos t) + 3 \sin t + 3(t \sin t + \cos t) \right]_0^{\pi/2} = -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4}$$

11. $\frac{\partial}{\partial y} \left[(1 + xy) e^{xy} \right] = 2xe^{xy} + x^2 y e^{xy} = \frac{\partial}{\partial x} \left[e^y + x^2 e^{xy} \right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x, y) = e^y + x^2 e^{xy}$ implies $f(x, y) = e^y + xe^{xy} + g(x)$ and then $f_x(x, y) = xye^{xy} + e^{xy} + g'(x) = (1 + xy)e^{xy} + g'(x)$. But $f_x(x, y) = (1 + xy)e^{xy}$, so $g'(x) = 0 \Rightarrow$ $g(x) = K$. Thus $f(x, y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and curl $\mathbf{F} = (0 - 0) \mathbf{i} - (0 - 0) \mathbf{j} + (\cos y - \cos y) \mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 13.5.4. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = \sin y$ implies $f(x, y, z) = x \sin y + g(y, z)$ and then $f_y(x, y, z) = x \cos y + g_y(y, z)$. But $f_y(x, y, z) = x \cos y$, so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Then $f(x, y, z) = x \sin y + h(z)$ implies $f_z(x, y, z) = h'(z)$. But $f_z(x, y, z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x, y, z) = x \sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y} \left(4x^3 y^2 - 2xy^3 \right) = 8x^3 y - 6xy^2 = \frac{\partial}{\partial x} \left(2x^4 y - 3x^2 y^2 + 4y^3 \right)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x, y) = x^4 y^2 - x^2 y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0, 1)$ and $t = 1$ corresponds to $(1, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$.

14. Here curl $\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x, y, z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$.

15.
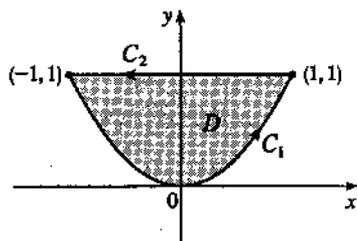
C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$; C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$. Then

$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1} (t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt$$

$$= \left[-\frac{1}{6}t^6\right]_{-1}^{1} + \left[\frac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(-x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right]dA = \iint_D (-2xy - 2xy)\,dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx = \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1}\,dx$$

$$= \int_{-1}^{1}(2x^5 - 2x)\,dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

16. $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right]dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx$

$$= \int_0^1 9x^2\,dx = 3x^3\big]_0^1 = 3$$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint_{x^2+y^2\le 4}\left[\frac{\partial}{\partial x}\left(-xy^2\right) - \frac{\partial}{\partial y}\left(x^2y\right)\right]dA$

$$= \iint_{x^2+y^2\le 4}(-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$$

18. $\operatorname{curl}\mathbf{F} = \left(0 - e^{-y}\cos z\right)\mathbf{i} - \left(e^{-z}\cos x - 0\right)\mathbf{j} + \left(0 - e^{-x}\cos y\right)\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,

$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F}$

$$= \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$

$$- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j} + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$

and

$$(\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} = \left[\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right.$$

$$+ \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$

$$- \left[\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right.$$

$$+ \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$

Hence

$$\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

$$= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right.$$

$$\left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}$$

$$+ \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right.$$

$$\left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$$

$$+ \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right.$$

$$\left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}$$

$$= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k}$$

$$= \operatorname{curl}(\mathbf{F}\times\mathbf{G})$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to

$$\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} \text{ to get } \int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right]dA = \iint_D 0\,dA = 0.$$

22. $\nabla^2(fg) = \dfrac{\partial^2(fg)}{\partial x^2} + \dfrac{\partial^2(fg)}{\partial y^2} + \dfrac{\partial^2(fg)}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right) \quad \text{(Product Rule)}$$

$$= \frac{\partial^2 f}{\partial x^2}g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2} \quad \text{(Product Rule)}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle\cdot\left\langle\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f\cdot\nabla g$$

[continued]

Another method: Using the rules in Exercises 11.6.33(b) and 13.5.23, we have

$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\nabla f + f\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$

$$= g\nabla^2 f + f\nabla^2 g + 2\nabla f \cdot \nabla g$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-f_x) - \frac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA$$
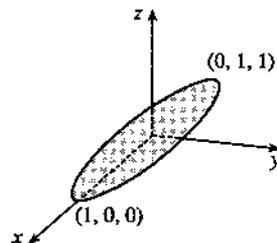
$$= -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 13.3.3.

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.

(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix}$$

$$= \left\langle -2y\csc^2 z - (-2y\csc^2 z),\, 0,\, 4xe^{2y} - 4xe^{2y} \right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

25. $z = f(x, y) = x^2 + y^2$ with $0 \leq x^2 + y^2 \leq 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\,dS = \iint_{x^2+y^2 \leq 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\,dA = \int_0^{2\pi} \int_0^2 r^3\sqrt{1 + 4r^2}\,dr\,d\theta$$

$$= \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

26. $z = f(x, y) = 4 + x + y$ with $0 \leq x^2 + y^2 \leq 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\,dS = \iint_{x^2+y^2 \leq 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3(4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

27. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (z - 2)\,dV = \iiint_E z\,dV - 2\iiint_E dV = m\bar{z} - 2\left(\tfrac{4}{3}\pi 2^3\right) = -\tfrac{64}{3}\pi.$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4\sin\phi\cos\theta\cos\phi\,\mathbf{i} - 4\sin\phi\sin\theta\,\mathbf{j} + 6\sin\phi\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$, and

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \left(16\sin^3\phi\cos\phi\cos^2\theta - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta\right) d\phi\,d\theta$$

$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi$$

28. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and

$\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \le 1} \left(-2x^3 - 2xy^2 + x^2 + y^2 \right) dA$$

$$= \int_0^1 \int_0^{2\pi} \left(-2r^3 \cos^3 \theta - 2r^3 \cos\theta \sin^2\theta + r^2 \right) r\,dr\,d\theta = \int_0^1 r^3 (2\pi)\,dr = \tfrac{\pi}{2}$$

29. Since curl $\mathbf{F} = \mathbf{0}$, $\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\cos^2 t \sin t + \sin^2 t \cos t \right) dt = \tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t \big]_0^{2\pi} = 0.$

30. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so

$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t \sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t \sin t}\,\mathbf{k}$, and

$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t \cos t$. Thus

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-16\cos^2 t \sin^2 t + 4\sin t \cos t \right) dt$

$= \left[-16\left(-\tfrac{1}{4}\sin t \cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t \right) + 2\sin^2 t \right]_0^{2\pi} = -4\pi.$

31. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_D (-1)\,dA = -(\text{area of } D) = -\tfrac{1}{2}$$

32. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi \int_0^1 (6r^3 + 8r)\,dr = 11\pi$

33. $\iiint_E \text{div } \mathbf{F}\,dV = \iiint_{x^2+y^2+z^2 \le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi,\theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi \cos^2\theta + \sin^3\phi \sin^2\theta + \sin\phi\cos^2\phi = \sin\phi$ and

$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi.$

34. Here we must use Equation 13.8.6 since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\,dS + \iiint_E \text{div } \mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so

$\text{div } \mathbf{F} = \nabla \cdot (|\mathbf{r}|^{-3}\mathbf{r}) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot (\nabla |\mathbf{r}|^{-3}) = |\mathbf{r}|^{-3}(3) + \mathbf{r} \cdot (-3|\mathbf{r}|^{-4})(\mathbf{r}|\mathbf{r}|^{-1}) = 0$. (Here we have

used Exercises 13.5.28(a) and 13.5.29(a).) And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1. Thus

$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = \iint_{S_1} dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi.$

35. Because curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x, y, z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, 3, 0) - f(0, 0, 2) = 0 - 4 = -4.$

36. Let C' be the circle with center at the origin and radius a as in the figure. Let D be

the region bounded by C and C'. Then D's positively oriented boundary is

$C \cup (-C')$. Hence by Green's Theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \sin^2 t - 2a \sin t}{a^2} (-a \sin t) \right.$$

$$\left. + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$
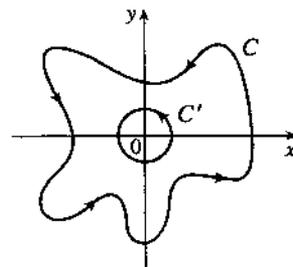
37. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3(\text{volume of } E) = 3(8 - 1) = 21.$

38. The stated conditions allow us to use the Divergence Theorem. Hence

$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0.$