Math 21a Final Exam
(Fall 2000)

1) __ 2) __ 3) __ 4) __ 5) __ 6) __ 7) __ 8) __ 9) __ 10) __ 11) __ : Total ______

Name: ________________________________

Section TF:

Allcock • Chen • Karigiannis • Knill • Liu • Rasmussen • Rogers • Taubes • Winter • Winters

Instructions:

- Print your name in the line above and circle the name of your section TF.

- **THERE ARE THREE PARTS TO THIS EXAM:**

A. All students answer Part A questions.
B. Only students in either the Regular or Physics sections answer Part B questions.
C. Only students in the BioChem sections answer Part C questions.

- Answers are to be written on the same page as the question. However, if more space is needed, use the back of the facing page. Use pages with unanswered questions (either those in Part C or in Part B, as the case may be) if additional space is needed.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.
- All problems will count the same amount.
- Neither unstaple nor remove pages from your exam booklet.

In agreeing to take this exam, you are implicitly accepting Harvard University's honor code.

PART A: The questions in this part of the exam should be answered by all students.

1. Integrate $y^{1/2}(x^2 + y^2)^{-3/2}$ over the region in the x-y plane where $-\infty < x < \infty$ and $1 \leq y < \infty$.

2. Integrate $\cos(z)\sin(z)\sin(y^2\cos(z))$ over the region in space where $0 \le x \le 1$, $x \le y \le 1$ and $0 \le z \le \pi/2$.

3. Write down, but do not evaluate, expressions in cartesian, then cylindrical, and finally spherical coordinates that compute the volume of the region in space where all of the following conditions are met: The distance to the origin is less than 5, $x \geq 0$, $y \geq 0$ and $z \geq 3$.

4. Find all of the extreme points of the function $x^2 + y^2 - 4x - 3y$ in the region of the plane where $x^2 + y^2 \leq 25$ and indicate those where the function takes its global maxima and minima in the region.

5. Find the points on the $z \geq 0$ portion of the surface $z^2 - x^2 - y^2 = 1$ which are closest to $(1, 1, 0)$.

6. Circle the number below which is closest to the value of $f(x, y, z) = x\ e^{-z^2 \sin(y)} + \sin(x^2 + z)$ at the point (.001, .002, .003)? You need not justify your answer.

a) 1/1000.

b) 0.

c) −1/250.

d) 1/250.

e) 3/500.

f) 249/250.

7. This question concern the surface, S, in space where $(\sqrt{x^2+y^2}-4)^2+z^2=1$.

a) Give the absolute value of the dot product of (5, 0, 0) with a unit length normal vector to S at the point (3, 4, 0).

b) Give the norm of the cross product of $(\sqrt{3}, 1, 0)$ with a unit length normal vector to S at the point $(1, \sqrt{3}, 1)$.

c) What is the maximum of the coordinate x on S?

d) The value of the coordinate x can be thought of as defining a function on S. This is the obvious function that assigns the number x to the point whose coordinates are (x, y, z). How many extreme points does this function have on S?

e) How many points in S are distance 6 from the origin?

PART B: The questions in this part of the exam are only for students in either the Regular or the Physics sections. Students in the BioChem sections do not answer questions in this part of the exam; they turn immediately to Part C of the exam.

9. Let S be the piece of the paraboloid given by $z = 1 - x^2 - y^2$ where $z \geq 0$. Let **n** denote the normal to S which points in the +z direction at (0, 0, 1). Let **F** denote the vector field

$$\mathbf{F} = (x + y \sin(z^2),\ y + x \sin(z^2),\ 1 - 2z).$$

   a) By parametrizing S, write down a 2-variable integral that computes the flux of **F** through S in the direction **n**.
   b) Find a surface S´ which is not S but is such that the fluxes of **F** through S´ and S have the same absolute value.
   c) What is the value of the integral that you wrote down for Part a?

10. Let $\mathbf{F} = (x + y \sin(z^2), y + x \sin(z^2), 1 - 2z)$.

a) Compute the path integral of $\mathbf{F}$ from $(1, 0, \sqrt{\pi/2})$ to $(1/\sqrt{2}, 1\sqrt{2}, \sqrt{\pi/2})$ in the increasing y direction along the circle where $x^2 + y^2 = 1$ and $z = \sqrt{\pi/2}$

b) Explain why $\mathbf{F}$ has zero path integral around every closed loop whose tangent vectors are all orthogonal to the z-axis.

c) Compute the absolute value of the line integral of $\mathbf{F}$ around the boundary of the square where $0 \le x \le 1$, $0 \le z \le \sqrt{\pi}$ and $y = 3$.

11. Write down, but do not evaluate a 2-variable integral that computes the surface area of the $x \geq 0$ part of the surface where $z^4 + x^4 + 2\,y^2\,x^2 + y^4 = 16$.

12. This question concerns soley those vector fields $\mathbf{v} = (a(x, y, z), b(x, y, z), c(x, y, z))$ in space which have both curl and divergence everywhere zero. Write the letter 'A' next to those statements below that are true for all of these vector fields; write the letter 'S' next to those statements that are true for some but not for all of these vector fields; and write the letter 'N' next to those statements that are never true for a vector field of this type. You need not justify your answers.

a) The functions a, b and c are constant on all of $\mathbb{R}^3$.

b) The function a has the property that $a_{xx} + a_{yy} + a_{zz} = 0$.

c) Where $z = 0$, the functions a, b and c obey $a = 0$, $b = 0$, and $c_z = 1$.

d) $\mathbf{v}$ is the gradient of a function.

e) $a_x = 0$, $b_y = 0$ and $c_z = 0$.

PART C: The questions in this part of the exam are only for students in the BioChem sections. Students in either the Regular or Physics sections do not answer questions in this part of the exam.

9. According to one source, the sensitivity of the SUDS test for HIV is 0.999 and the specificity of this test is 0.996. In the nation of Botswana, about one in three adults is infected with HIV.
   a) What are the positive and negative predictive values of the SUDS test for the nation of Botswana?
   b) The sensitivity and positive predictive value of a test represent information that medical practitioners need to know. In practice, which of these two can actually be measured? Explain your reasoning in a brief sentence.
   c) Would a medical practitioner rather know the sensitivity or positive predictive value of an HIV test? Explain your reasoning in a brief sentence.

10. A quarter of the diners at a certain restaurant order a fish dinner, half order a meat dinner and the remainder order only a green salad. Two thirds of those that order a fish dinner also order a green salad as a first course, while one third of those that order a meat dinner order a green salad as as a first course.

a) What is the probability that a diner at this restaurant will order a salad?
b) Suppose that you see a diner eating a green salad. What is the probability that th diner will only have a salad? What is the probability that the diner will also have fish?
c) If the average menu price for a fish dinner is \$8, the average for a meat dinner is \$10 and that for a green salad is \$2, what is the average amount (excluding tip and tax) spent by a diner at this restaurant?

11. On average, three phone calls in four to a computer help hot line receive busy signals. Assume that this statistic is independent of the number of times that any given person calls the provider. Also, suppose that the help hotline receives an average of 1000 calls per hour, and that the number of calls received in any given hour has a Poisson distribution.

a) Write down an expression that gives the probability that your first ten calls to the help hotline get busy signals and your eleventh call gets through. Do not evaluate your expression.

b) Write down an expression that gives the probability that at least your first ten calls to the help hotline get busy signals. Do not evaluate your expression.

c) Write down, but do not evaluate, an expression for the probability that the hot line receives 3000 calls in a given two hour period.

Do not justify your answer.

12. Suppose that the following probabilities are normally distributed with the indicated mean, μ, and standard deviation σ:

i) The average daily Boston temperature in January: $\mu = -2°$ C and $\sigma = 4°$ C.

ii) The average daily Boston temperature in July: $\mu = 25°$ C and $\sigma = 8°$ C.

iii) The average daily Boston temperature in May: $\mu = 15°$ C and $\sigma = 6°$ C.

Based on this data, re-order the following list so that the probability of occurrence decreases going down the list. Do not justify your answer.

a) The average temperature on July 15 is greater than or equal to 32° C.

b) The average temperature on January 8 is less than – 5° C.

c) The average temperature on May 15 is greater than 20° C.

d) The average temperatures on July 15 is between 25° C and 33° C.