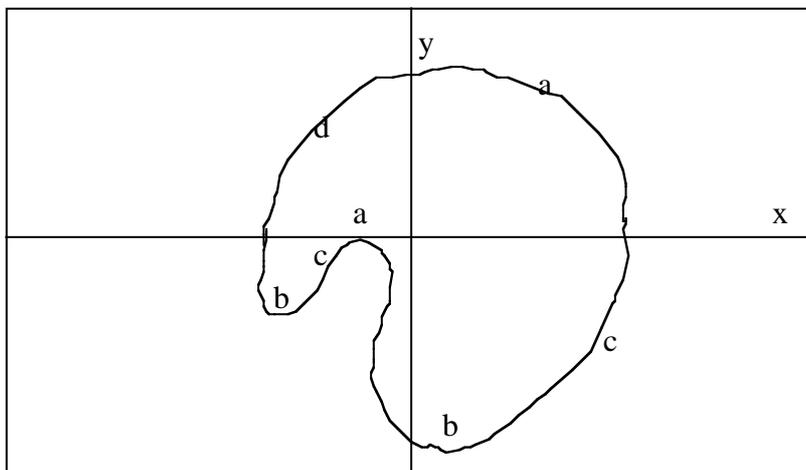# Math 21a Practice Final Exam Ansers

PART A

1.

Note: No points are marked for e.

2. a) $\int_0^{1/\sqrt{2}}\left(\int_y^{(1-y^2)^{1/2}}\left(\int_z^{(1-x^2-y^2)^{1/2}}xyzdz\right)dx\right)dy$.

b) $\int_0^1\left(\int_0^{\pi/4}\left(\int_0^{(1-r^2)^{1/2}}r^2z\cos\theta\sin\theta dz\right)d\theta\right)rdr$.

c) $\int_0^1\left(\int_0^{\pi/2}\left(\int_0^{\pi/4}\rho^3\cos\phi\sin^2\phi\cos\theta\sin\theta d\theta\right)\sin\phi d\phi\right)\rho^3d\rho$.

3. a) $(2, 2, 1)$ is one such vector. The vector $(2, 7, 0)$ is another.

b) $-7x + 2y + 10z = 0$.

c) $L(x, y, z) = 6 + \frac{1}{9}(-7x + 2y + 10z)$.

d) $\frac{5}{9\sqrt{3}}$.

4. $\frac{1}{96}$.

5. The square of the norm of the gradient is $\frac{1}{64}x^2 + \frac{1}{4}y^2 + 4z^2$. This has its maximum on the boundary at $(0, 0, \pm 1)$.

6. One such equation is $t \to (1 + 3t, -2t, t)$.

7. $\frac{2}{15}$

8. a) A; if $\nabla u = 0$, then u is constant and u_{xx} and u_{yy} are both zero.
   b) S; for example $u = y^2/2$ solves the equation as does $u = x^2/2$.
   c) A; At a global maximum, $u_{xx} + u_{yy} \leq 0$.
   d) N; the origin would then be a local maximum so $u_{xx} + u_{yy} \leq 0$ there.
   e) S; this is the case for $\frac{1}{4}(x^2 + y^2)$, but not the case for $\frac{1}{4}(x^2 + y^2) - 3000\,x$.

PART B

9. a) The flux through B is bigger. Here is why: First, $\iint_A \mathbf{F} \bullet \mathbf{n} \, dS + \iint_B \mathbf{F} \bullet \mathbf{n} \, dS =$
    $\iiint_V \text{div}(\mathbf{F}) \, dV$ where V is the region between A and B with $\mathbf{n}$ being the outward pointing
    normal from V. Thus, $\mathbf{n} \bullet (0, 0, 1) > 0$ on B and $\mathbf{n} \bullet (0, 0, 1) < 0$ on A. Meanwhile $\text{div}(\mathbf{F}) = 2$
    $+ \, 2 \, z > 0$ on V. Therefore, the A flux in question (computed with $-\mathbf{n}$) must be less than the
    B flux computed with $\mathbf{n}$.
    b) The flux of curl($\mathbf{F}$) is the same through both since they both have the same boundary (use
    Stokes' theorem).
    c) Since the disk where $z = 0$ and $x^2 + y^2 \le 1$ has the same boundary as the surface under
    consideration, one can just as well compute the flux through this disk. The normal to this
    disk is $(0, 0, 1)$ and curl($\mathbf{F}$) where $z = 0$ is $(y, x, 0)$ and so the flux of curl($\mathbf{F}$) is zero.

10. a) Parametrize the surface by (u, v) where $0 \le u \le 1$ and $0 \le v \le 2\pi$ according to the rule that
    sends $(u, v) \rightarrow (u, u/2 \cos(v), u/2 \sin(v))$. Then, the surface area is $\int_0^{2\pi} \left(\int_0^1 \frac{\sqrt{5}}{4} u \, du \right) dv$.

    b) $-\int_0^{2\pi} \left(\int_0^1 \frac{1}{4} u^2 \, du \right) dv$

11. a) $\mathbf{v} = (z, -z, 0)$
    b) Not possible since a unit radius disk, S, in this plane has normal $(1, 0, 0)$ whose dot
    product with $(1, 1, 0)$ is 1. Thus, the flux of the hypothetical vector field through S would
    equal π and so, by Stokes' theorem, its line integral over the boundary circle would not be
    zero.
    c) $\mathbf{v} = (2x, 0, 0)$.
    d) Not possible by the divergence theorem: The flux through any closed surface would equal
    twice the volume of the enclosed region.

12. a) By Green's theorem, $\mathbf{v} = (0, \frac{1}{3} x^3)$.
    b) $\int_0^{2\pi} \frac{1}{3} \cos^4 t \, dt = \frac{1}{4} \pi$.

PART C

9. a) $\mu = 56$ and $\sigma = 16.8$.

   b) $\displaystyle\sum_{m=62}^{80}\binom{80}{m}(.7)^m(.3)^{80-m}$.

   c) $\displaystyle\int_{62}^{\infty}\frac{1}{\sqrt{2\pi}16.8}\exp(-(x-56)^2/33.6)\,dx$

10. a) $\frac{26}{27}$.

    b) $\frac{1}{54}$.

    c) $\frac{1}{9}$.

11. a) $(12.5)^k\,\frac{1}{k!}e^{-12.5}$.

    b) Since $\mu = 25$ and $\sigma = 25$, this is 3/5 standard deviations from the mean.

    c) The probability of 40 deaths in one 4-year span is $(25)^{40}\,\frac{1}{40!}e^{-25}$. The probability of 40 deaths in two consecutive four year spans is the square of this last number.

12. a) 10.3 percent.

    b) Sensitivity $= \frac{.013}{.103} \sim 0.13$. $PV^+ = \frac{.013}{.1} = 0.13$.

    c) 0.1.

Part D

9. a) $\frac{16}{125}$.

   b) $\frac{61}{125}$.

   c) 5. This is the mean for the probability density on the positive integers that assigns to the integer n the probability, $\left(\frac{4}{5}\right)^{n-1}\frac{1}{5}$, of Bernoulli going to jail on the n'th day

   d) $\frac{80}{369}$.

10. Since $G(x) = G(x)^1$, the assertion is true for $n = 1$. Now, suppose that the assertion holds for all integers between 1 and $n - 1$. To show that it then holds for integer n, use the fact that $G(x + y) = G(x)G(y)$ to write $G(nx) = G(x + (n-1)x) = G(x)G((n-1)x)$. Next, invoke the induction hypothesis that $G((n-1)x) = G(x)^{n-1}$ to equate $G(nx)$ with $G(x)G(x)^{n-1}$ which is the desired $G(x)^n$.

11. a) The program must write each two element set once and only once.

    b) Here is an algorithm: For each positive integer in turn starting with 2, list that integer with each of the positive integers that are strictly less than it. Also, number the whole list consecutively from 1.

For those interested in a more 'computer language' algorithm, consider the following translation:

Step 1: Let m = 2.
Step 2: Let k = 1.
Step 3: Let n = 1.
Step 4: Print 'k → (n, m)'.
Step 5: Let k = k + 1.
Step 6: If n = m – 1, go to Step 9.
Step 7: Let n = n + 1.
Step 8: Go to Step 4.
Step 9: Let m = m + 1.
Step 10: Go to Step 3.

Here are the first six outputs:

$1 \to \{1, 2\}, 2 \to \{1, 3\}, 3 \to \{2, 3\}, 4 \to \{1, 4\}, 5 \to \{2, 4\}, 6 \to \{3, 4\}.$

c) For each integer N, let A_N denote the collection of n-element sets of distinct, positive integers with largest integer N. Note that A_N is a set with $\frac{(N-1)!}{(n-1)!(N-n)!}$ elements. This understood, the collection $a = \{A_N\}_{1 \leq N < \infty}$ is the collection of all n-element sets of distinct positive integers, here exhibited as a countable union of finite sets. Thus, assuming the claim, the collection a is itself countable.

12. a) Since the integral of f over R^2 must equal 1, $c = \frac{1}{18}$.
    b) $\frac{1}{270}$.
    c) $\frac{43}{12}$.