(1) Find all critical points of the function $f(x, y) = x^3 + y^2 - 6xy + 6x + 3y - 2$ and for each point, determine whether it is a local maximum, a local minimum, or a saddle point.

$f_x(x, y) = 3x^2 - 6y + 6 = 0$

$f_y(x, y) = 2y - 6x + 3 = 0$

$\rightarrow y = 3x - 1.5$

$\rightarrow 3x^2 - 6(3x - 1.5) + 6 = 3x^2 - 18x + 15 = 0$

$\rightarrow x^2 - 6x + 5 = 0 \rightarrow (x-5)(x-1) = 0$

$\rightarrow x = 5$ or $x = 1 \rightarrow (5, 13.5)$ or $(1, 1.5)$

are the critical points

$f_{xx}(x, y) = 6x$

$f_{xy}(x, y) = -6$

$f_{yy}(x, y) = 2$

$D(a, b) = (6a)(2) - (-6)^2 = 12a - 36$

$D(5, 13.5) > 0$ and $f_{xx}(5, 13.5) > 0$, so $(5, 13.5)$ is a local min.

$D(1, 1.5) < 0$ so $(1, 1.5)$ is a saddlepoint

(2) Find the maximum and minimum values of the function $f(x, y) = x - y + 3z$ on the surface given by the equation $x^2 + 2y^2 + 2z^2 = 24$.

$$\nabla f = \lambda \nabla g :$$

$$1 = 2\lambda x \qquad -1 = 4\lambda y \qquad 3 = 4\lambda z$$

$$\frac{1}{4\lambda^2} + \frac{2}{16\lambda^2} + \frac{2 \cdot 9}{16\lambda^2} = 24$$

$$\to 2 + 1 + 9 = 24 \cdot 8\lambda^2 \to \lambda^2 = \frac{1}{16} \to \lambda = \pm\frac{1}{4}$$

Case $\lambda = \frac{1}{4}$: $x = 2,\ y = -1,\ z = 3 \to f = 12$

Case $\lambda = -\frac{1}{4}$: $x = -2,\ y = 1,\ z = -3 \to f = -12$

Thus the max of f on this surface is 12 (occurring at $(2,-1,3)$), and the min of f on this surface is -12 (occurring at $(-2,1,-3)$).

(3) Two planes are given by the equations $2x + y + z = 2$ and $x - y - 3z = 4$. Find the point on their line of intersection that is closest to the origin.

$$\begin{aligned} 2x + y + z &= 2 \\ x - y - 3z &= 4 \\ \hline 3x - 2z &= 6 \end{aligned}$$

$x = t \rightarrow z = 1.5t - 3$ $\quad$ $t - y - 3(1.5t - 3) = 4$

$\rightarrow y = -3.5t + 5$

So the line of intersection is parametrized by

$x = t$

$y = -3.5t + 5$

$z = 1.5t - 3$

Minimize $t^2 + (-3.5t+5)^2 + (1.5t-3)^2 = d(t)$

$d'(t) = 2t + 2(-3.5t+5)(-3.5) + 2(1.5t-3)(1.5)$

$= 31t - 44 = 0$

So $t = 44/31$ and the point is

$\left(44/31, \ 1/31, \ -27/31\right)$

(4) Given the line $(x, y, z) = (5-2t, 2+t, -3+2t)$ in $\mathbb{R}^3$ and the point $(9, 5, -2)$,

a) find an equation for the plane containing both the line and the point.

b) find the shortest distance from the point (6, 1, 0) to the plane you just found.

a) a point on the plane is $(5,2,-3)$. So get the vector $(4,3,1)$.

$(4,3,1) \times (-2,1,2) = (5,-10,10)$

So the plane is $5x - 10y + 10z = 5(5) - 10(2) + 10(-3)$
$= -25$

$\rightarrow x - 2y + 2z = -5$

b) Want $(6,1,0) + t(1,-2,2)$ in the plane:

$(6+t) - 2(1-2t) + 2(0+2t) = -5$

$\rightarrow 9t = -9 \rightarrow t = -1$

So the distance is $\sqrt{(-1)^2 + (2)^2 + (-2)^2} = \sqrt{1+4+4} = 3$

(5) What is the average value of the function $f(x, y) = x^2$ over the unit disk in $\mathbb{R}^2$, i.e., the region $x^2 + y^2 \leq 1$?

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r\cos\theta)^2 r\, dr\, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos^2\theta \left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4\pi} \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= \frac{1}{4\pi}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{4\pi}\left[\pi + 0 - 0 - 0\right] = \frac{\pi}{4\pi} = \frac{1}{4}$$

(6) What is the directional derivative of the function $f(x, y, z) = x^3 + y^2 + z$ in the direction perpendicular to the surface $xy + 5yz - 3z^2 = 5$ at the point $(-1,2,1)$?
(Choose the direction with positive z-component.)

Direction perpendicular means find the gradiant at the point:

$$(y)\vec{i} + (x+5z)\vec{j} + (5y-6z)\vec{k}$$

$$\rightarrow (2,4,4) \longrightarrow \frac{1}{\sqrt{4+16+16}}(2,4,4) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \vec{u}$$

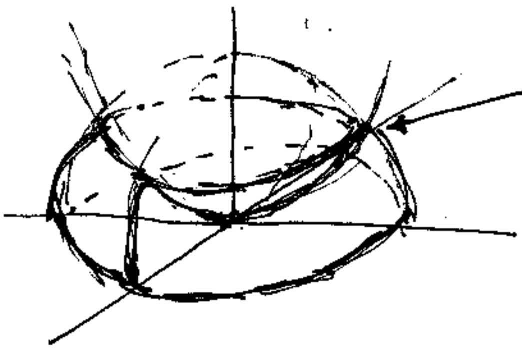
Directional Derivative: $D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$

$$= \left(3x^2, 2y, 1\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= x^2 + \frac{4}{3}y + \frac{2}{3}$$

at $(-1,2,1)$, this is $\frac{13}{3}$

(7) Find the volume of the solid bounded below by $z = 0$, above by $z^2 = x^2 + y^2$, and on the side by $x^2 + y^2 + z^2 = 2$.



Circle of intersection at $z^2 + z^2 = 2$: $z = 1$.

In cylindrical coordinates:

$$z = 0, \quad z^2 = r^2, \quad z^2 + r^2 = 2$$

$$\text{Solid} = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ z^2 \le r^2 \le 2 - z^2\}$$

$$\text{Volume} = \iiint_{\text{solid}} 1\, dV = \int_0^{2\pi} \int_0^1 \int_z^{\sqrt{2-z^2}} r\, dr\, dz\, d\theta$$

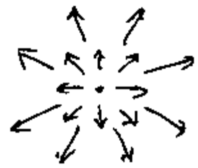
$$= \int_0^{2\pi} \int_0^1 \left(\frac{2-z^2}{2} - \frac{z^2}{2} \right) dz\, d\theta = \int_0^{2\pi} \int_0^1 (1 - z^2)\, dz\, d\theta$$

$$= \int_0^{2\pi} \left(z - \frac{z^3}{3} \right)\Big|_0^1 d\theta = \int_0^{2\pi} \left(1 - \frac{1}{3}\right) d\theta = \frac{4\pi}{3}$$

(8) Which of the following vector fields are not conservative? Briefly explain your reasoning.

(a)

(b)



(c)

(d)

(a) constant vectors, so $\vec{F} = a\vec{i} + b\vec{j}$, and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 0$ so it's conservative

(b) looks like $\vec{F} = x\vec{i} + y\vec{j}$, here again $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 0$ so again conservative

(c) it's easy to find closed paths which give non-zero work integrals, such as the one shown, so it's not conservative

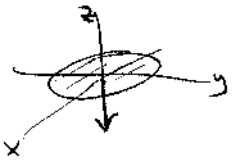
(d) looks like $\vec{F} = \langle -x, 0 \rangle$, but tilted. The tilt doesn't change being conservative or not, and $\vec{F}$ is conservative, so (d) is likely conservative as well.

(9) Water is flowing down a vertical cylindrical pipe of radius 2 inches. The velocity vector field of the water is given by $\mathbf{v} = (r^2 - 4)\mathbf{k}$ where r is the distance in inches from the center of the pipe. How much water flows out of the bottom of the pipe in 3 seconds?

This is asking for a flux integral $\iint_S \vec{F} \cdot ds$

where $\vec{F} = \langle 0, 0, \underbrace{(x^2+y^2)}_{r^2} - 4 \rangle$

and S is the circular region $x^2+y^2 \leq 2$ in the x-y plane with normal vector pointing downwards



so $S: \langle x, y, 0 \rangle \quad x^2+y^2 \leq 2$

$(\vec{r}_x \times \vec{r}_y) = \langle 0, 0, 1 \rangle$, but we want it downward pointing so use $(\vec{r}_y \times \vec{r}_x) = \langle 0, 0, -1 \rangle$

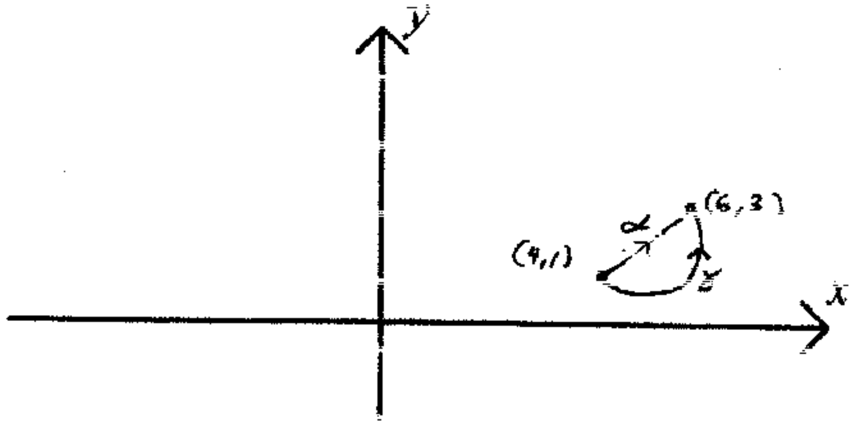
so flux integral $= \iint_S \left(-(x^2+y^2) + 4 \right) dx\,dy$

or in polar coordinates $\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\,dr\,d\theta$

$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{4} r^4\right)\Big|_0^2 d\theta = 2\pi(8-4) = 8\pi$$

(10) Calculate the work integral $\int_\gamma \mathbf{F}\cdot d\mathbf{x}$ for the curve γ shown, a semicircle from the point (4,1) to the point (6,3) where $\mathbf{F}(x, y) = (x + y)\mathbf{i} + (3x - 2y)\mathbf{j}$.

(Hint: It's possible to find the value of this integral without parametrizing the semicircle.)



Remember Green's Theorem:

$$\int_\gamma F\cdot d\mathbf{r} = \iint_D (3-1)\,dA + \int_\alpha F\cdot d\mathbf{r}$$

$$= 2\cdot\tfrac{1}{2}(\pi)(\sqrt{2})^2 + \int_\alpha F\cdot d\mathbf{r}$$

$\alpha: r(t) = (4+2t)\vec{i} + (1+2t)\vec{j}$

$0 \le t \le 1$

$r'(t) = 2\vec{i} + 2\vec{j}$

$$= 2\pi + \int_0^1 \left[2(5+4t) + 2(10+2t)\right] dt$$

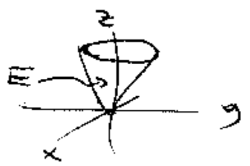
$$= 2\pi + \int_0^1 [30 + 12t]\,dt = 2\pi + \left[30t + 6t^2\right]_0^1$$

$$= 2\pi + [30+6] = 2\pi + 36$$

(11) Compute the flux of the vector field $F(x,y,z) = (e^{y^2+z^2}, y^2+z^2, e^{x^2+y^2})$ across a portion of the cone with equation $4(x^2+y^2) = 9z^2$ lying between $z=0$ and $z=2$ oriented with a downward normal.

That vector field look suspicious – smell a divergence theorem problem! Div F $= (0, 2y, 0)$, much simpler!

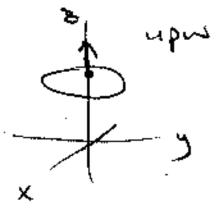
Now $\iiint_E (\text{Div } F)\, dV = \iiint_E 2y \, dV$ where E is the interior region of the cone bounded above by $z=2$ (above cone, below $z=2$)



since E is completely symmetric with regards to the y variable, then $\iiint_E 2y \, dV = 0$

so $\iint_{S_1} \vec{F} \cdot dS + \iint_{S_2} \vec{F} \cdot dS = 0$

where S_1 is the surface given, and S_2 is the top, circular region, of the cone, normal pointing upwards. This region has $z=2$, so $4(x^2+y^2) = 9(2^2)$



$\Rightarrow x^2+y^2 = 9$ is the boundary of this circle, which we parametrize as $\langle x, y, 2 \rangle$ with $x^2+y^2 \leq 9$,

and it's easy to calculate $\iint_{S_2} \vec{F} \cdot ds$

here the upward pointing normal $(\vec{r}_x \times \vec{r}_y) = \langle 0, 0, 1 \rangle$,

and we get $\iint_{S_2} \vec{F} \cdot dS = \iint_R e^{(x^2+y^2)} \, dx\, dy \quad R: x^2+y^2 \leq 9$

switch to polar: $= \int_0^{2\pi} \int_0^3 r e^{r^2} \, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2} e^{r^2} \Big|_0^3 \right) d\theta$

$= 2\pi \left(\frac{1}{2} e^9 - \frac{1}{2} \right) = \pi e^9 - \pi$, so since

$\iint_{S_2} \vec{F} \cdot ds = - \iint_{S_1} \vec{F} \cdot ds$, then the original flux integral is $\pi - \pi e^9$

(12) Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y^2x, y^2x, y^2x)$. Let D be the portion of the solid ball $x^2 + y^2 + z^2 \leq 9$ which lies in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 9$). Set up, **but do not evaluate**, a triple integral <u>in spherical coordinates</u> which gives the flux of $\mathbf{F}$ out of the boundary of the region D.

Use the divergence theorem - flux integral over boundary of $D = \iiint_D \operatorname{div}(\vec{F})\, dV$,

where D in spherical coordinates is just

$0 \leq \phi \leq \pi/2 \quad 0 \leq \theta \leq \pi/2 \quad 0 \leq \rho \leq 3$

$\operatorname{div}(\vec{F}) = y^2 + 2xy$, in spherical coordinates

where $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$,

this is equal to $\rho^2\sin^2\theta\sin^2\phi + 2\rho^2\sin\theta\cos\theta\sin^2\phi$

$= \rho^2\sin^2\phi\,(\sin^2\theta + 2\sin\theta\cos\theta)$

so flux of $\vec{F}$ out of boundary of D is

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^2\sin^2\phi\,(\sin^2\theta + 2\sin\theta\cos\theta)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$