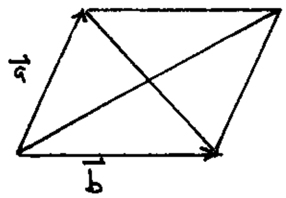
Math 2/a Final Exam - Answers   May 16^{th}, 2002

(1)(a)



let $\vec{a}, \vec{b}$ be vectors along the two sides of the parallelogram as shown, then note that the diagonals of the parallelogram are equal to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$

We are told that the diagonals are perpendicular, so $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$, however $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$

$$=(\vec{a}\cdot\vec{a})+(\vec{b}\cdot\vec{a})-(\vec{a}\cdot\vec{b})-(\vec{b}\cdot\vec{b})$$

since $\vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}$ then this simplifies as $\vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{b}$ which $=0$,

so $\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}$, which we also know is $|\vec{a}|^2=|\vec{b}|^2$ or so the (length of $\vec{a}$) = (length of $\vec{b}$), i.e. the two sides, (and thus all sides) are the same length

(b) Now consider $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = (\vec{a}\times\vec{a})+(\vec{b}\times\vec{a})-(\vec{a}\times\vec{b})-(\vec{b}\times\vec{b})$
we know $\vec{a}\times\vec{a}$ and $\vec{b}\times\vec{b}=0$ (basic cross product property) and $-(\vec{a}\times\vec{b})=\vec{b}\times\vec{a}$, so the cross product of the diagonals is just equal to $2(\vec{b}\times\vec{a})$, where $\vec{a},\vec{b}$ are the vectors representing the two sides as drawn above. We know $|\vec{b}\times\vec{a}|$ = area of parallelogram (basic cross product property - just check $|\vec{b}\times\vec{a}|=|\vec{b}||\vec{a}|\sin\theta$, θ = angle between $\vec{b},\vec{a}$), so the magnitude of the cross product of the diagonals equals twice the area of the parallelogram, and since we never used the fact that the diagonals are perpendicular, then this relationship holds for any given parallelogram.

(2)(a) the line that lies in both planes is simply their intersection. so $2x+3y-z=2$ can be written as $z=+2x+3y-2$ and the other plane is $z=\frac{1}{2}x-y+2$, setting these equal yields $2x+3y-2=\frac{1}{2}x-y+2$, or $4y=-\frac{3}{2}x+4$, so $y=-\frac{3}{8}x+1$,

and $z=2x+3y-2=2x+3\left(-\frac{3}{8}x+1\right)-2$
$=\frac{7}{8}x+1$

so then the line is the set of points $\langle x,y,z\rangle$ with $y=-\frac{3}{8}x+1$, $z=\frac{7}{8}x+1$, i.e. a set of parametric

(2) (a) continued... equations is just $x = t$, $y = -\frac{3}{8}t + 1$, $z = \frac{7}{8}t + 1$

(b) the normal vectors are just $\langle 2,3,-1\rangle$ and $\langle -1,2,2\rangle$ respectively, so the angle between them comes from the dot product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$,

so $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{-2+6-2}{(\sqrt{4+9+1})(\sqrt{1+2+2})}\right) = \cos^{-1}\left(\frac{2}{3\sqrt{14}}\right)$

(3) (a) the information tells us that the gradient vector at the point $(4,5,6)$ is in direction $\langle 4,0,-3\rangle$, and has magnitude $= 7$, so $\nabla f(4,5,6) = \frac{\langle 4,0,-3\rangle}{\sqrt{4^2+0^2+3^2}}$ times $7 = \langle \frac{28}{5}, 0, \frac{-21}{5}\rangle$

Now, the rate of increase of $f(x,y,z)$ in the $\langle -1,2,-2\rangle$ direction is $\nabla f(4,5,6)\cdot\vec{u}$, where $\vec{u} = \frac{\langle -1,2,-2\rangle}{\sqrt{1^2+2^2+2^2}} = \langle -\frac{1}{3},$

$= \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$

so this equals $\langle \frac{28}{5}, 0, -\frac{21}{5}\rangle \cdot \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$

$= \frac{-28+42}{15} = \frac{14}{15}$

(b) Aha, recall that the gradient vector at a point is perpendicular to the level curve through that point (or perpendicular to the level surface is the function has 3 variables). We know the gradient vector at the point $(4,5,6)$ is $\langle \frac{28}{5}, 0, -\frac{21}{5}\rangle$,

so the equation for a plane gives $\frac{28}{5}x + 0y + \left(\frac{-21}{5}\right)z =$ a constant

plug in $(4,5,6)$, which is a point on the plane,

so $\left(\frac{28}{5}\right)4 + 0 + \left(\frac{-21}{5}\right)6 = \frac{112}{5} - \frac{126}{5} = -\frac{14}{5}$,

so the equation of the plane is $\frac{28}{5}x - \frac{21}{5}z = -14$

or simply $28x - 21z = -14$

or $4x - 3z = -2$!

(It would have been easier to start with the vector $\langle +4,0,-3\rangle$ from part (a), which we know is parallel to the gradient of $f(x,y,z)$ at $(4,5,6)$)

(4) (a) $f(x,y) = x^2+y^2+2y-1$, so $\nabla f = \langle 2x, 2y+2\rangle$

$\nabla f = \langle 0,0\rangle$ when $x=0$, $y=-1$ and that's it (only critical pt.)

check $D = f_{xx}\cdot f_{yy} - (f_{xy})^2$ at $(0,0)$: $2\cdot 2 - 0 > 0$,

and $f_{xx} = 2 > 0$, so it's a local minimum. In fact

by rewriting $f(x,y) = x^2 + (y+1)^2 - 2$, then clearly the

minimum at $(0,-1)$ is an absolute minimum for $f(x,y)$

(b) constraint curve $y = x^2+1$ needs to be rewritten

as $g(x,y) = y - x^2 - 1 = 0$ (or $g(x,y) = y - x^2 = 1$, etc.)

then set $\nabla f = \lambda \nabla g$: $\langle 2x, 2y+2\rangle = \lambda \langle -2x, 1\rangle$

then $2x = -\lambda 2x$, so either $\lambda = -1$, or $x=0$

(critical to spot the $x=0$ possibility!). If $\lambda = -1$,

then $2y+2 = \lambda = -1$, so $y = -\frac{3}{2}$, which is impossible

given that $y = x^2+1 > 0$, so $x=0$, in which

case $y = x^2+1 = 1$, so the extreme points of

$f(x,y)$ on $y=x^2+1$ consists just of the point $(0,1)$

(c) Now parametrize $y=x^2+1$, simply take $x=t$, then

$y = x^2+1 = t^2+1$, so $\vec{r}(t) = \langle t, t^2+1\rangle$ $-\infty < t < \infty$

then $f(x,y) = f(t, t^2+1) = (t)^2 + (t^2+1)^2 + 2(t^2+1) - 1$

$= t^4 + 5t^2 + 2$

and $f'(t) = 4t^3 + 10t = 2t(2t^2+5)$ $2t^2+5$ has no

real roots, so $t=0$ is the only real root of $f'(t)=0$

and this leads to the same $(0,1)$ point as in part (b)

(d) use the $f(x,y) = x^2 + (y+1)^2 - 2$ form of $f(x,y)$ to see

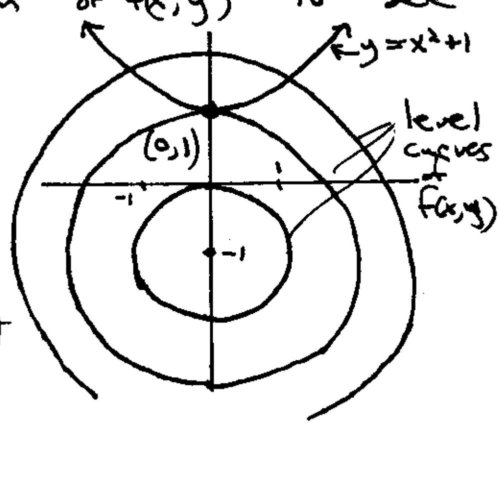
that the level curves of $f(x,y)$

form concentric circles around $(0,-1)$

The $y=x^2+1$ constraint curve is

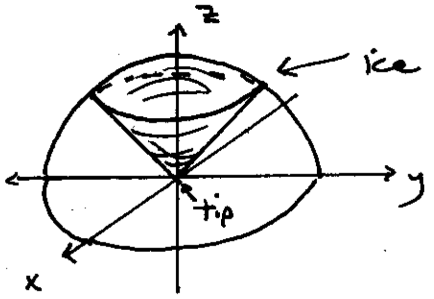
just a parabola. The point $(0,1)$ is

the only place where the constraint

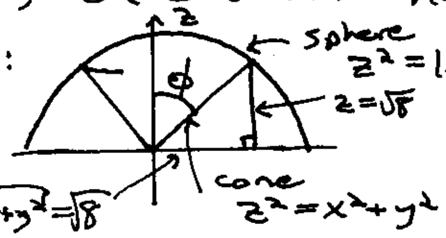


(4) (d) continued ... curve is tangent to a level curve. As the level curves increase in value as they increase in size, this one point $(0,1)$, the extreme point of $f(x,y)$ on $y=x^2+1$, must be a minimum

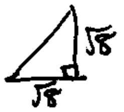
(5)



density $\sigma = k(10-d)$ where d is d is $\underbrace{\text{distance}}$ from tip, i.e. $d=\sqrt{x^2+y^2+z^2}$. Hmm, ... in <u>spherical</u> coordinates this is just ρ. Hmm... the top of the cone is described by the equation for a <u>sphere</u> of radius 4, $x^2+y^2+z^2=4^2=16$ ... we should try setting up the integral in spherical coordinates then the ice cream cone region can be described as $0 \le \rho \le 4$, $0 \le \theta \le 2\pi$. How about ϕ? Look at a cross-section:



intersect where
$x^2+y^2=16-x^2-y^2$
or $x^2+y^2=8=z^2$

which makes the right triangle  a 45° triangle, ($\frac{\pi}{4}$ radians)

so $0 \le \phi \le \pi/4$,

so the integral is $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 k(10-\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= k\int_0^{2\pi}\int_0^{\pi/4}\left(\frac{10\rho^3}{3}-\frac{\rho^4}{4}\right)\Big|_0^4 \sin\phi\,d\phi\,d\theta = k\left(\frac{640}{3}-64\right)\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi/4}\,d\theta$$

$$= 2\pi k\left(\frac{640}{3}-64\right)\left(-\frac{1}{\sqrt{2}}-(-1)\right) = \frac{448}{3}k\left(2-\sqrt{2}\right)\pi$$

(6) (a) Since $\vec{F}$ is a conservative vector field, $\vec{F} = \nabla f$, then we know that the fundamental theorem of line integrals says that $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$ where A, B are the beginning and ending points of the curve C. but since the curve C is presumed to be on a level surface of $f(x,y,z)$, then $f(B) = f(A) = k$, no matter where A and B are on the surface M, so $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = k - k = 0$

(b) Part (a) should tip us off to be on guard for conservative vector fields! Check: if $\vec{F} = P\vec{i} + Q\vec{j}$ $= \langle -e^y \sin(x), e^y \cos(x) \rangle$ then $\frac{\partial Q}{\partial x} = -e^y \sin(x) = \frac{\partial P}{\partial y} \ldots$

aha! Find a potential function:

$\int -e^y \sin(x)\, dx = e^y \cos(x) + g(y)$, some $g(y)$ function of y alone, and $\frac{\partial}{\partial y}\left(e^y \cos(x) + g(y)\right) = e^y \cos(x) + g'(y)$ should $= e^y \cos(x)$,

so in fact $\vec{F} = \nabla f$ with $f = e^y \cos(x)$.

Now find endpoints of curve C: when $t = 0$ (start) $\vec{r}(0) = \langle \pi - \pi, \pi \cdot 0 \rangle = \langle 0, 0 \rangle$, at $t = \pi$ (end point)

$\vec{r}(\pi) = \langle \pi - \pi\cos(\pi), \pi \sin(\pi) \rangle = \langle 2\pi, 0 \rangle$,

so $\int_C \vec{F} \cdot d\vec{r} = f(2\pi, 0) - f(0,0) = e^0 \cos(2\pi) - e^0 \cos(0)$

$= 0$!

(Note – it's certainly possible to get 0 as a result even if you're calculating a line integral over a path that's not closed!)

(7) Green's Theorem tells us that $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ equals $\oint_{\partial R} \vec{F} \cdot d\vec{r}$, where $\vec{F} = P\vec{i} + Q\vec{j}$, and ∂R is the boundary of a region R in the xy-plane. We're told to consider $\vec{F} = \langle 0, x^3 y\rangle$. Aha! Here $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 y - 0 = 3x^2 y$, the integrand in the line integral we are supposed to calculate, so $\iint_R (3x^2 y)\, dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$, where $\vec{r}(t)$ needs to parametrize the boundary of the elliptical region.

so ∂R: $x^2 + \frac{y^2}{4} = 1$, the hint tells us to check out $x = \cos(t)$, $y = 2\sin(t)$, i.e. $\vec{r}(t) = \langle \cos t, 2\sin t\rangle$ clearly to hit the whole boundary $0 \leq t \leq 2\pi$

So... $\iint_R (3x^2 y)\, dA = \oint_C F \cdot d\vec{r} = \int_0^{2\pi} \langle 0, x^3 y\rangle \cdot \vec{r}\,'(t)\, dt$

$$= \int_0^{2\pi} \langle 0, (\cos t)^3 (2\sin t)\rangle \cdot \langle -\sin t, 2\cos t\rangle\, dt$$

$$= \int_0^{2\pi} 4\cos^4 t \sin t\, dt = \left(-\frac{4}{5}\cos^5 t\right)\Big|_0^{2\pi} = 0$$

(8) Stokes' Theorem says $\iint_S (\text{curl}\, \vec{F}) \cdot dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$

so here, to use Stokes' we need to convert the given line integral into $\iint_S (\text{curl}\, \vec{F}) \cdot dS$, where we can pick any surface S with boundary given by $\langle x, y, z\rangle = \langle x, y, x^2 - y^2\rangle$ with $x^2 + y^2 = 1$

(8) continued Pick the first "obvious" surface at hand – the surface of the graph of $z = f(x,y) = x^2 - y^2$, then we can use the usual parametrization $\langle x, y, f(x,y)\rangle$ over the circular region $x^2+y^2 \le 1$, so $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle = \langle x, y, x^2-y^2\rangle$, $\vec{r}_x \times \vec{r}_y = \langle -2x, 2y, 1\rangle$. Check this is upward pointing → at $(0,0)$ $\vec{r}_x \times \vec{r}_y = \langle 0,0,1\rangle$ (upward pointing is correct so that the boundary is oriented counter-clockwise when looking down the z-axis).

Next if $\vec{F} = \langle x^2+z^2, y, z\rangle$ then

$$\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \langle 0-0, 2z-0, 0-0\rangle = \langle 0, 2z, 0\rangle,$$ and on

$S: \langle x, y, x^2-y^2\rangle$ then $\text{curl}\,\vec{F} = \langle 0, 2(x^2-y^2), 0\rangle$

so the line integral $\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F})\cdot dS$

$= \iint_R \langle 0, 2x^2-2y^2, 0\rangle \cdot \langle -2x, 2y, 1\rangle \, dA$, where

R is the circular region $x^2+y^2 \le 1$, so $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ as $-1 \le x \le 1$, so the integral

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (4x^2y - 4y^3)\,dy\,dx = \int_{-1}^{1} \left(2x^2y^2 - y^4\right)\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left[\left(2x^2(1-x^2) - (1-x^2)^2\right) - \left(2x^2(1-x^2) - (1-x^2)^2\right)\right] dx = \int_{-1}^{1} 0\,dx = 0$$

There are a number of ways of setting this integral up, but obviously you should always end up with 0!

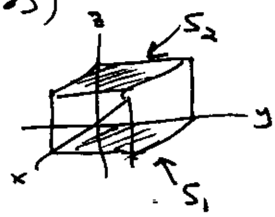
(9) So we're supposed to find $\iiint_R \operatorname{div}(\vec{E})\,dV$ where $\vec{E} = \langle x(1-x)\log(1+xyz),\ y(1-y)\tan(xyz),\ z(1-z)e^{xyz} \rangle$ over R: unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. Two observations: finding $\operatorname{div}(\vec{E})$ is extremely ugly, and there are those weird looking "$x(1-x)$" "$y(1-y)$" "$z(1-z)$" factors in $\vec{E}$.

Thinking hard... do we have another way to compute $\iiint_R \operatorname{div}(\vec{E})\,dV$? Sure! The divergence theorem says that this equals $\iint_S \vec{E} \cdot dS$, where

S is the surface equal to the "boundary" of R, i.e. the 6 square faces of the cube (with outward pointing normals), so $\iiint_R \operatorname{div}(\vec{E})\,dV = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

(each integral $= \iint_{S_i} \vec{E}\,dS$)

where the S_i are the six faces of the cube



So we calculate: for $S_1 = \langle x, y, 0 \rangle$

$0 \le x \le 1$, $0 \le y \le 1$ $\quad \vec{r}_x \times \vec{r}_y = \langle 0, 0, +1 \rangle$ which points the opposite direction, from what we need, so take $\vec{r}_y \times \vec{r}_x = \langle 0, 0, -1 \rangle$, now calculate $\vec{E}$ on this S_1 face

Yea!, this is where the magic happens for on S_1,

$\vec{E} = \langle *_1, *_2, 0 \rangle$ ($*_i$ is some function of x, y, z) since $z = 0$

so $\iint_{S_1} \vec{E} \cdot dS = \iint_{S_1} \langle *_1, *_2, 0 \rangle \cdot \langle 0, 0, -1 \rangle\, dA = \iint_{S_1} 0\, dA = 0$

On the top face, S_2, $z = 1$, and again $\vec{E} = \langle *_1, *_2, 0 \rangle$, and again $\iint_{S_2} \vec{E} \cdot dS = \iint_{S_2} \langle *_1, *_2, 0 \rangle \cdot \langle 0, 0, 1 \rangle\, dA = \iint_{S_2} 0\, dA = 0$

This same phenomenon occurs on each of the other four

(9) continued faces as well: 

on S_3 where $y=0$ $\vec{E} = \langle *_1, 0, *_2 \rangle$, and

$$\iint_{S_3} \vec{E} \cdot dS = \iint_{S_3} \langle *_1, 0, *_2 \rangle \cdot \langle 0,-1,0 \rangle \, dA = \iint_{S_3} 0 \, dA = 0$$

on S_4 $\vec{E} = \langle *_1, 0, *_2 \rangle$ (different $*_i$'s each time, of course)

and one calculates $\iint_{S_4} \vec{E} \cdot dS = \iint_{S_4} \langle *_1, 0, *_2 \rangle \cdot \langle 0, 1, 0 \rangle \, dA$
$= 0$

i.e. the outward pointing normal for each face has a ± 1 in just the one component where $\vec{E}$ is zero on that face, and 0's in the other two components where $\vec{E}$ has non-zero components, so the dot product is always 0,

so $\iiint_R \operatorname{div}(E) \, dV$ = total charge $= 0+0+0+0+0+0 = 0$

(10) A Regular/Biochem: (numbered as question 11 on biochem)

(a) so $f(x,y) = g(x^2+y^2)$, then by chain rule we have $\frac{\partial f}{\partial x} = g'(x^2+y^2)(2x)$ and $\frac{\partial f}{\partial y} = g'(x^2+y^2)(2y)$

so $y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} = y \cdot g'(x^2+y^2)(2x) - x \cdot g'(x^2+y^2)(2y)$
$= 2xy \, g'(x^2+y^2) - 2xy \, g'(x^2+y^2) = 0$

so $f(x,y)$ is in fact a solution to this PDE

(b) Note in this particular PDE $\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}$,
we can take $c = 3$,
and $l = 2\pi$ (since boundary conditions are $u(0,t) = u(2\pi, t) = 0$)

so general solution is $u(x,t) = \left(A \cos\left(\frac{3n}{2} t\right) + B \sin\left(\frac{3nt}{2}\right)\right) \sin\left(\frac{nx}{2}\right)$

now $u(x,0) = 0$ is an initial condition, so subbing in $t=0$ in our solution, we get

(10) A continued $u(x,0) = \left(A\cos(0) + B\sin(0)\right)\sin\left(\frac{nx}{2}\right)$

$= A\sin\left(\frac{nx}{2}\right)$. so to be $=0$, implies

A must equal 0.

Now that's it, so the general solution is simply

$u(x,t) = B\sin\left(\frac{3nt}{2}\right)\sin\left(\frac{nx}{2}\right)$ for B a constant, n an integer.

When it asks for one example, we just need to choose particular values for n and B, such as 1:

$u(x,t) = \sin\left(\frac{3t}{2}\right)\sin\left(\frac{x}{2}\right)$ or $n=2$, $B = 5331.9$:

$u(x,t) = 5331.9\sin(3t)\sin(x)$. That's it!

(10) B Physics section:

Assume $z > R$, then $\vec{E} = E(z)\vec{n}$ (z is distance from center)

so $\oiint \vec{E}\cdot\vec{n}\,dS = \oiint E(z)\,\vec{n}\cdot\vec{n}\,dS = E(z)\oiint dS = E(z)\,4\pi z^2$,

and since this equals $\frac{Q}{\varepsilon_0}$, then $E(z) = \frac{Q}{\varepsilon_0 4\pi z^2}$

Assume $z < R$, then $Q' = \frac{4}{3}\pi z^3 \rho$ since not all of the sphere is enclosed

if $\frac{z}{R} = r$, then $Q' = r^3 Q$ since ρ is constant

Thus $E(z) = \left(\frac{z}{R}\right)^3 \frac{Q}{\varepsilon_0 4\pi z^2} = \frac{Qz}{\varepsilon_0 4\pi R^3}$

Now a point charge is always $E(z) = \frac{Q}{\varepsilon_0 4\pi z^2}$ since you can't "get inside" a point charge where $R=0$