Math 21a Midterm 1 Solutions - Spring 2000

1. (a) Setting $x = t$ we get two equations in two variables: $2y - z = 15 - 2t$ and $3y - 3z = 32 - 5t$. Solving these equations gives the parametrization of the intersection of the two planes:

$$x = t, \qquad y = \frac{13-t}{3}, \qquad z = \frac{4t-19}{3}.$$

(b) We want to know the angle between the vectors $\vec{u} = (0, 0, 1)$ and $\vec{v} = (1, -1/3, 4/3)$. This is:

$$\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\frac{4/3}{\sqrt{1+1/9+16/9}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{26}}\right).$$

(c) The shortest vector between a plane and a point will be parallel to the normal vector to the plane. Thus this vector is

$$t(2,2,-1) + (-1,1,3) \text{ for some } t.$$

Since this vector has to lie in the plane, we can find the value of t:

$$2(2t-1) + 2(2t+1) - (-t+3) = 15 \Rightarrow t = 2.$$

Thus the point on the plane closest to the point $(-1,1,3)$ is $(3,5,1)$.

2. (a) We can parameterize this line by $x = 2 - t$, $y = 3t$, $z = 7t$.

(b) The vector $\overrightarrow{NM}$ is $(-1,-3,1)$ and the vector $\overrightarrow{NP}$ is $(-4,-5,2)$. Thus the normal vector to the plane containing M, N, and P is $\overrightarrow{NM} \times \overrightarrow{NP} = (-1,-2,-7)$. Since $-1(2) - 2(0) - 7(0) = -2$, the equation for the plane is $x + 2y + 7z = 2$.

(c) The area of the triangle MNP is $.5|\overrightarrow{NM} \times \overrightarrow{NP}| = .5|(-1,-2,-7)| = .5\sqrt{1+4+49} = .5\sqrt{54}$.

(d) Following the hint, let's find the perpendicular height. This is the shortest distance from the plane containing MNP to Q. As in 1c), we find that the vector we want to measure to is $(0,1,0)$. Thus the volume is $(1/3)(.5\sqrt{54})(\sqrt{54}) = 9$

3. (a) $v(t) = p'(t) = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t, -10e^{-t})$.

(b)

$$\int_0^\infty \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2 + (-10e^{-t})^2}\,dt$$

$$= \int_0^\infty \sqrt{2e^{-2t} + 100e^{-2t}}\,dt$$

$$= \sqrt{102}\int_0^\infty e^{-t}\,dt = \sqrt{102}.$$

4.

$\vec{u}\cdot\vec{v} = -2$	$\frac{\vec{r}\cdot\vec{u}}{\vec{r}} = \sqrt{5}$	$\vec{u}\times\vec{v} = (8,4,0)$
$3\vec{u} - 2\vec{v} = \langle -5, 10, 7\rangle$	$\frac{\vec{r}\cdot\vec{u}}{\lvert\vec{r}\rvert^2}\vec{r} = (1,0,2)$	$\lvert\vec{u}\times 2\vec{v}\rvert = 8\sqrt{5}$
$\lVert -2r\rVert = 2\sqrt{5}$	$\cos^{-1}\left(\frac{\vec{r}\cdot\vec{u}}{\lvert\vec{r}\rvert\lvert\vec{u}\rvert}\right) = \cos^{-1}\left(\sqrt{5/14}\right)$	$\lvert 3\vec{r}\cdot(\vec{u}\times 2\vec{v})\rvert = 48$

5. The midpoints of the sides of the parallelogram have coordinates $(\vec{a}+\vec{b})/2$, $(\vec{b}+\vec{c})/2$, $(\vec{c}+\vec{d})/2$, and $(\vec{a}+\vec{d})/2$. So the vectors connecting these midpoints are (in order around the quadrilateral they form): $(\vec{c}-\vec{a})/2$, $(\vec{d}-\vec{b})/2$, $(\vec{a}-\vec{c})/2$, and $(\vec{b}-\vec{d})/2$. Since the first and third, and second and fourth are multiples of each other, opposite sides of the quadrilateral formed by the midpoints are parallel. Therefore the midpoints form a parallelogram.

6. (a) The point P is (x, y, z), so this is represented by the vector $\vec{u} = (x, y, z)$. We almost want the projection of $\vec{u}$ onto the line L, but that's not quite it. The shortest vector from P to L is actually $\vec{u}$ minus its projection onto L. Thus this vector is

$$\vec{u} - \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \vec{u} - \frac{y+z}{2}(0,1,1) = \left(x, \frac{y-z}{2}, \frac{z-y}{2}\right).$$

Thus the (shortest) distance from P to L is

$$\sqrt{x^2 + \frac{(y-z)^2}{2}}.$$

(b) $x^2 + \frac{(y-z)^2}{2} = 1$.

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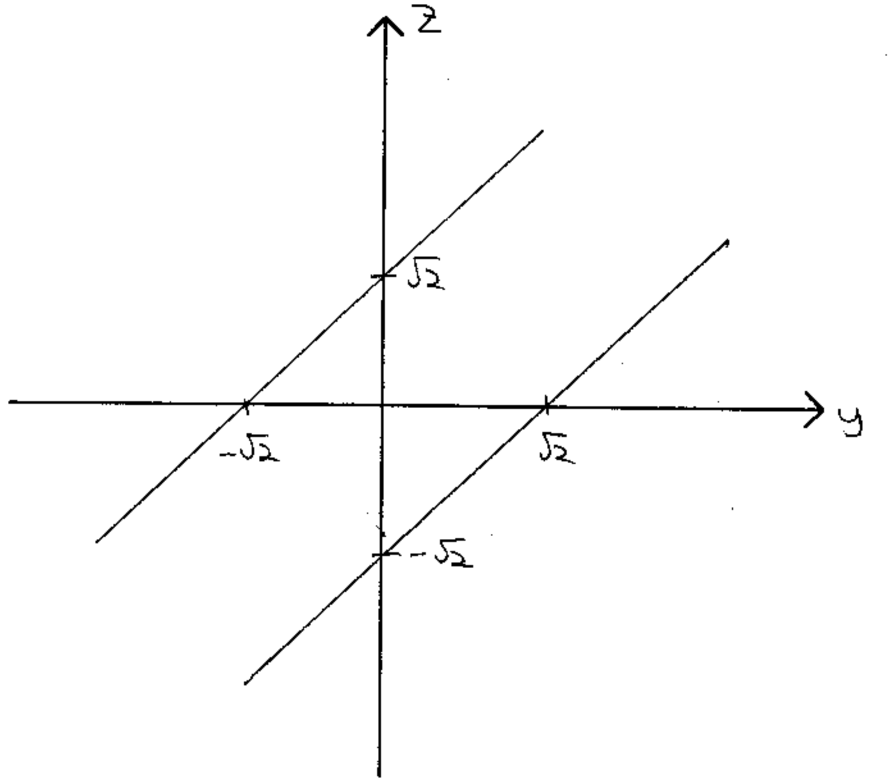
(c) $x = 0 \Rightarrow (y-z)^2 = 2$. A graph of this (where the y-axis is horizontal and the z-axis is vertical) consists of two parallel lines, each with slope 1. One of these intercepts the z-axis at $(0, \sqrt{2})$ and the other has an intercept of $(0, -\sqrt{2})$.

(d) $y = 0 \Rightarrow x^2 + .5z^2 = 1$. A graph of this (where the x-axis is horizontal and the z-axis is vertical) consists of an ellipse with intercepts $(\pm 1, 0)$ and $(0, \pm\sqrt{2})$.

c) Draw an $x = 0$ cross section of the cylinder. Label and number your axes.



d) Draw a $y = 0$ cross section of the cylinder. Label and number your axes.