Math 21a Hourly 2 with Solutions
(Fall 1999)

1. Consider the surface where the function $g(x, y, z) = y^2 - 3xy + zx + 2x - z$ is zero.
   a) Find the tangent plane to this surface at (-1, 0, -1).
   b) Find a point P on this surface where the tangent plane is given by the equation $x = y$. Explain your reasoning.
   c) Find the best linear approximation to this function $g(x, y, z)$ at the point (-1, 0, -1).
   d) Find a direction (a unit vector) in which the directional derivative of g at (-1, 0, -1) is zero.

Solution to problem 1:

a) The tangent plane is the plane through the given point which is orthogonal to the gradient. Here, $\nabla g = (-3y + z + 2, 2y - 3x, x - 1)$. At (-1, 0, -1), one has $\nabla g = (1, 3, -2)$. Thus, the tangent plane consists of the points (x, y, z) which obey $x + 3y - 2z - 1 = 0$.

b) A normal vector to the plane where $x = y$ is the vector (1, -1, 0). This is proportional to ∇g when $x = 1$ and $z = y + 1$. Furthermore, a point $(1, y, z = y + 1)$ has $g = 0$ only when $y^2 - 3y + 2 = 0$, which has $y = 1$ or $y = 2$. Thus, P = (1, 1, 2) or P = (1, 2, 3). However, the second point must be rejected since the tangent plane at the point P is given by the equation $x = y$. So P = (1, 1, 2).

c) $L(x, y, z) = x + 3y - 2z - 1$ has the same value as g at (-1, 0, -1) and the same gradient.

d) Any vector **u** which obeys $\nabla g \cdot \mathbf{u} = 0$. Thus, $\mathbf{u} = (-3a + 2b, a, b)/(10a^2 + 5b^2 - 12ab)^{1/2}$ where both a and b are not zero. For instance, $\mathbf{u} = \dfrac{(-3, 1, 0)}{\sqrt{10}}$.

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2. Suppose that the moon is modeled by the ball where $x^2 + y^2 + z^2 \leq 1$. If the temperature of a point (x, y, z) in or on the moon at a particular time is given by the function

$$T(x, y, z) = 50(1 - x^2 - y^2 - z^2) + 10(\sqrt{3}\,x + z)$$

then find:
   a) The (x, y, z) coordinates of the hottest of the points on the surface of the moon.
   b) The (x, y, z) coordinates of the points in or on the moon where temperature is greatest.
   c) The (x, y, z) coordinates of the points in or on the moon where the temperature is least.

In all cases, make sure you justify your conclusions.

Solution to problem 2:

a) The hottest point on the surface is $(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$. Temp = 20 degrees.

b) The hottest point inside or on the surface is $(\frac{\sqrt{3}}{10}, 0, \frac{1}{10})$.

This is inside the moon and the temp = 52 degrees.

c) The coldest point is $(-\frac{\sqrt{3}}{2}, 0, -\frac{1}{2})$. Temp = -20 degrees.

The extreme points on the surface are obtained by solving the Lagrange multiplier equations for those points where $x^2 + y^2 + z^2 = 1$ and $\nabla T = \lambda \nabla g$ where $g(x, y, z) = x^2 + y^2 + z^2$.

Here, $\nabla T = 10(\sqrt{3}, 0, 1) - 100(x, y, z)$ and $\nabla g = (2x, 2y, 2z)$.

The extreme point inside is obtained by solving for the points where $x^2 + y^2 + z^2 < 1$ and $\nabla T = \mathbf{0}$.

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3. The positions of a rat and a snail on the x-y plane are given, respectively by

$$\mathbf{r}(t) = (1 + t)\,\mathbf{i} + (t^2 + t)\mathbf{j} \quad \text{and} \quad \mathbf{s}(t) = \cos(t)\,\mathbf{i} + (t^3 - t)\,\mathbf{j},$$

where t is time measured in seconds.

a) Give the velocity and acceleration vectors for the rat at $t = 0$.

b) Give the velocity and acceleration vectors for the snail at $t = 0$.

c) The temperature of any point (x, y) on the plane is position dependent and thus given by a function, $T(x, y)$, measured in degrees. In particular, note that the temperature at any given point doesn't change with time. However, as the rat and snail are moving, they feel the temperature change. For example, at $t = 0$ the rat feels the temperature increase at the rate of 3 degrees per second, while the snail feels the temperature decrease at the rate of 1 degree per second. With the preceding understood, compute ∇T at the point (1, 0).

Solution to problem 3:

a) $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}''(0) = 2\mathbf{j}$.

b) $\mathbf{s}'(0) = -\,\mathbf{j}$ and $\mathbf{s}''(0) = -\,\mathbf{i}$.

c) Write $\nabla T = a\mathbf{i} + b\mathbf{j}$. Then we are told that $\nabla T \cdot \mathbf{r}'(0) = a + b = 3$ and $\nabla T \cdot \mathbf{s}'(0) = -b = -1$. Thus, $b = 1$ and $a = 2$ and $\nabla T = 2\mathbf{i} + \mathbf{j}$ at (1, 0).

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4. Let $f(x, y) = x^2 y - 4xy + y^3/3$.

a) Find all critical points of f.

b) Identify the points found in Part a as local maxima, minima or saddle points. Justify your answers here.

c) Find the directions of maximum increase and decrease of f at (1, -3).

d) Find the tangent vector to the level curve of f through (2, 4).

Solution to problem 4:

a) The critical points occur where $\nabla f = (2xy - 4y, x^2 - 4x + y^2) = 0$. These are $(2, \pm 2)$, (0, 0) and (4, 0).

b) (2, 2) is a local minimum, (2, -2) is a local maximum, (0, 0) is a saddle, (4, 0) is a saddle. The 2nd derivative test establishes these assertions since the matrix f'' of 2nd derivatives has $\det(f'') > 0$ and $\text{trace}(f'') > 0$ at (2, 2), and $\det(f'') > 0$ and $\text{trace}(f'') < 0$ at (2, -2), and $\det(f'') < 0$ at (0, 0) and (4, 0).

c) The direction of maximum increase is $\frac{(1, 1)}{\sqrt{2}}$. That of maximum decrease is $\frac{(-1, -1)}{\sqrt{2}}$.

d) Any vector of the form $(a, 0)$ with $a \neq 0$.

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5. Use the best linear approximation to $f(x, y) = e^x y^{1/2}$ at a convenient point to estimate the value of f at $(0.1, 25.3)$.

Solution to problem 5:

The best linear approximation to f at (0, 25) is $L(x, y) = 5 + 5x + \frac{1}{10}(y - 25)$.

Using L to estimate f gives 5 + .5 + .03 = 5.53.

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6. Integrate the function $f(x, y) = x^3 y$ over the region where $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 1$.

Solution to problem 6:

After doing the y integration, one is left with integrating $\frac{1}{2}x^3(1 - x^2)$ between 0 and 1.

The integral is $\frac{1}{24}$.

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7. Calculate the integral $\iint_R 2e^{x^2}\, dA$ where R is the triangle where $0 \leq y \leq 1$ and $y \leq x \leq 1$. Thus, the vertices of R are (0, 0), (1, 0) and (1, 1) in the x-y plane.

Solution to problem 7:

Do the y integral first. (There is no closed form expression for the x integral if that one is done first.) The range for the y integral is from $y = 0$ to $y = x$. The resulting x integral is for the function $2xe^{x^2}$ with the range going from $x = 0$ to $x = 1$. Changing variables to $u = x^2$ shows that this is the same as the integral of e^u from $u = 0$ to $u = 1$, which is $e - 1$.