

**Question 1.** (15 points total)

Suppose you have a suitcase with dimensions  $h$  = height,  $w$  = width,  $l$  = length that you're about to check in at the airport. When you check the suitcase, its dimensions are  $h = 2$ ,  $w = 1$  and  $l = 4$  (all measured in feet), but it's a bizarre suitcase whose dimensions are all changing with time. Suppose at the instant you check the suitcase  $\frac{dh}{dt} = \frac{dw}{dt} = 0.1$  feet/second, and that  $\frac{dl}{dt} = -0.3$  feet/second.

(a) (5 points) What is the rate of change of the volume of the suitcase when you check it in?

Volume =  $l \cdot w \cdot h = V(l, w, h)$



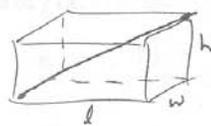
so 
$$\frac{dV}{dt} = \frac{\partial V}{\partial l} \cdot \frac{dl}{dt} + \frac{\partial V}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$
$$= wh \cdot (-.3) + lh \cdot (.1) + lw \cdot (.1)$$
$$= 1 \cdot 2 \cdot (-.3) + 4 \cdot 2 \cdot (.1) + 4 \cdot 1 \cdot (.1)$$
$$= -.6 + .8 + .4 = .6 \text{ cubic ft./sec.}$$

(b) (5 points) What is the rate of change of the surface area of the suitcase at that same time?

Surface Area =  $2wh + 2wl + 2lh = S(w, h, l)$

same set up 
$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial S}{\partial h} \cdot \frac{dh}{dt} + \frac{\partial S}{\partial l} \cdot \frac{dl}{dt}$$
$$= 2(h+l) \cdot (.1) + 2(w+l) \cdot (.1) + 2(w+h) \cdot (-.3)$$
$$= 2 \cdot 6 \cdot (.1) + 2 \cdot (5) \cdot (.1) + 2 \cdot (3) \cdot (-.3)$$
$$= 1.2 + 1.0 + (-1.8)$$
$$= 0.4 \text{ square ft./second}$$

(c) (5 points) Is the length of the suitcase's main interior diagonal increasing or decreasing when you check it?



main diagonal =  $\sqrt{l^2 + h^2 + w^2}$

just check  $l^2 + h^2 + w^2$  (incr. or dec.)  
 $\Rightarrow \sqrt{\quad}$  is too

is  $2l \frac{dl}{dt} + 2h \frac{dh}{dt} + 2w \frac{dw}{dt} > \text{ or } < 0$

or  $4 \cdot (-.3) + 2 \cdot (.1) + 1 \cdot (.1) = -1.2 + .2 + .1 < 0$

so decreasing

**Question 2** (16 points total)

Let  $f(x, y) = x^2 + xy^2$  and let point  $P = (-1, 2)$ .

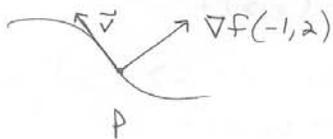
(a) (4 points) In what direction does the function  $f(x, y)$  increase most rapidly at the point  $P$ ? Give a unit vector  $\vec{u}$  for your answer.

$$\nabla f = \langle 2x + y^2, 2xy \rangle$$
$$\text{so } \nabla f(-1, 2) = \langle -2 + 4, -2 \cdot 2 \rangle = \langle 2, -4 \rangle$$
$$\text{then } \vec{u} = \frac{1}{\sqrt{2^2 + 4^2}} \langle 2, -4 \rangle = \frac{1}{\sqrt{20}} \langle 2, -4 \rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$$

(b) (4 points) What is the directional derivative  $D_{\vec{u}}f(-1, 2)$ ? (where  $\vec{u}$  is unit vector from part (a)).

$$\text{This is just } |\nabla f(-1, 2)| = \sqrt{20}$$
$$\text{(or } D_{\vec{u}}f(-1, 2) = \nabla f(-1, 2) \cdot \vec{u} = \sqrt{20} \text{)}$$

(c) (4 points) Find a vector  $\vec{v}$  tangent to the level curve of  $f(x, y)$  through the point  $P$ .



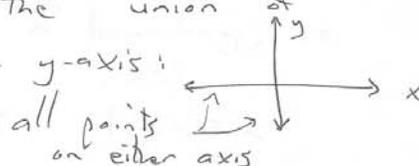
so  $\vec{v} \cdot \langle 2, -4 \rangle = 0$  (perp to  $\nabla f(-1, 2)$  is tangent to level curve as  $\nabla f$  is  $\perp$  to level curve). Take any such  $\vec{v}$ , such as  $\langle 4, 2 \rangle$  or  $\langle 2, 1 \rangle$  or  $\langle 1, \frac{1}{2} \rangle$

(d) (4 points) Give a description of all the points in the  $xy$ -plane at which there is no increase or decrease in the value of  $f(x, y)$  in the  $\langle 0, 1 \rangle$  direction.

$$\text{i.e. } D_{\langle 0, 1 \rangle} f(x, y) = 0. \text{ This is just } \frac{\partial f}{\partial y} = 2xy$$

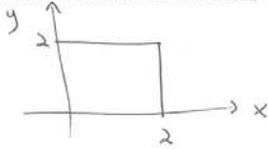
for this to be 0, either  $x=0$  or  $y=0$ ,

which is equivalent to the union of the  $x$  axis along with the  $y$ -axis:



**Question 3.** (15 points total)

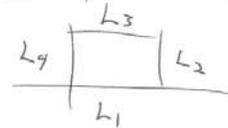
Consider the function  $f(x, y) = x^2 - 2xy + \frac{1}{2}y^2 + 2x$ . Find the absolute minimum and maximum values of this function on the square region with vertices  $(0,0)$ ,  $(2,0)$ ,  $(0,2)$  and  $(2,2)$  (where the region includes both the square's boundary and interior points). Find both the minimum and maximum values, and the points where these min and max values occur.



So start with critical points  
 for  $f(x, y)$ :  $\nabla f = \langle 2x - 2y + 2, -2x + y \rangle$   
 when is  $\nabla f = \langle 0, 0 \rangle \dots -2x + y = 0$ , from  
 so  $y = 2x$ ,  
 so sub back into  $2x - 2y + 2$   
 get  $2x - 2(2x) + 2 = 0 \rightarrow -2x + 2 = 0$   
 so  $x = 1$ , then  $y = 2$   
 only critical point.

at this point  $f(1, 2) = 1 - 4 + \frac{1}{2}4 + 2 = 1$

Now check min/max on boundary:



on  $L_1: \langle x, 0 \rangle \quad 0 \leq x \leq 2$

$$f(x, 0) = x^2 + 2x \quad \frac{df}{dx} = 2x + 2 = 0 \text{ when } x = -1, \text{ out of range}$$

on  $L_2: \langle 2, y \rangle \quad 0 \leq y \leq 2$

$$f(2, y) = 4 - 4y + \frac{1}{2}y^2 + 4 \quad \frac{df}{dy} = -4 + y = 0 \text{ when } y = 4, \text{ out of range}$$

on  $L_3: \langle x, 2 \rangle \quad 0 \leq x \leq 2$

$$f(x, 2) = x^2 - 4x + 2 + 2x \quad \frac{df}{dx} = 2x - 2 = 0 \text{ when } x = 1 \text{ already found } (1, 2)$$

on  $L_4: \langle 0, y \rangle \quad 0 \leq y \leq 2$

$$f(0, y) = \frac{1}{2}y^2 \quad \frac{df}{dy} = y = 0 \text{ when } y = 0, \text{ i.e. at } (0, 0), \rightarrow \text{will check}$$

Now check  $f(x, y)$  at boundaries of boundary, i.e.

at $(0, 0)$	$(0, 2)$	$(2, 0)$	$(2, 2)$
$f(0, 0) = 0$	$f(0, 2) = 2$	$f(2, 0) = 8$	$f(2, 2) = 2$

So... max at  $(2, 0)$  value = 8, min at  $(0, 0)$  value = 0

**Question 4.** (20 points total)

Find the coordinates of the point (or points) on the surface  $z^2 = x^2 + y^2 + 8$  that are closest to the plane

$x + y + 2z = 0$ . To do this you can use the formula  $\frac{|ax_1 + by_1 + cz_1|}{\sqrt{a^2 + b^2 + c^2}}$  for the distance of a point  $(x_1, y_1, z_1)$  to

the plane  $ax + by + cz = 0$ . Note that minimizing  $w^2$  is equivalent to minimizing  $|w|$ .

So given  $(x, y, z)$  on  $g(x, y, z) = z^2 - x^2 - y^2 = 8$   
minimize distance  $\frac{|ax + by + cz|}{\sqrt{a^2 + b^2 + c^2}}$  to plane with  
 $a = 1, b = 1, c = 2$

i.e. distance =  $\frac{|x + y + 2z|}{\sqrt{6}}$

equivalent to minimizing  $(x + y + 2z)^2 = M(x, y, z)$

so set  $\nabla g = \lambda \nabla M$  or

$$\langle -2x, -2y, 2z \rangle = \lambda \langle 2(x + y + 2z), 2(x + y + 2z), 4(x + y + 2z) \rangle$$

so ①  $-x = \lambda(x + y + 2z)$

②  $-y = \lambda(x + y + 2z)$

③  $z = 2\lambda(x + y + 2z)$

and ④  $g(x, y, z) = 8$

or  $z^2 = x^2 + y^2 + 8$

if  $\lambda = 0$  then  $x = y = z = 0$ , which doesn't satisfy ④

so consider  $\lambda \neq 0$  then from ① and ②  $-x = -y$ ,  
i.e.  $x = y$ ,

and from ① and ③ since  $-x = \lambda(x + y + 2z)$   
then  $z = 2 \cdot \lambda(x + y + 2z) = -2x$ .

so  $y = x, z = -2x$ , and in ④  $(-2x)^2 = x^2 + x^2 + 8 = 2x^2 + 8$

so  $4x^2 = 2x^2 + 8$ , or  $2x^2 = 8$

$\Rightarrow x = \pm 2$ ,

so possible points are

$(+2, +2, -4)$  and  $(-2, -2, 4)$

distances  $\frac{|x + y + 2z|}{\sqrt{6}}$  are both  $= \frac{4}{\sqrt{6}}$  or  $\frac{2}{3}\sqrt{6}$

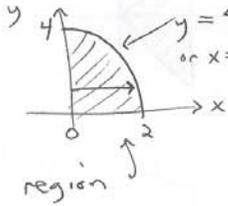
to check its a minimum, take any other point on surface,  
like  $(0, 0, \sqrt{8}) \rightarrow$  distance  $= \frac{|2\sqrt{8}|}{\sqrt{6}} = \frac{4\sqrt{2}}{\sqrt{6}} > \frac{4}{\sqrt{6}}$

so... these are the closest points

Question 5. (20 points total)

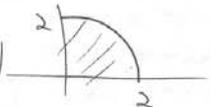
(a) (10 points) Evaluate the double integral  $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy dx$ .

can't do this one as it stands, so reverse order...  
 Now try  $\int \int \frac{xe^{2y}}{4-y} dx dy$



$$\begin{aligned} \text{so get } \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx dy &= \int_0^4 \frac{x^2}{2} \frac{e^{2y}}{4-y} \Big|_{x=0}^{\sqrt{4-y}} dy \\ &= \int_0^4 \frac{(\sqrt{4-y})^2}{2} \frac{e^{2y}}{4-y} dy = \int_0^4 \frac{1}{2} e^{2y} dy = \frac{1}{4} e^{2y} \Big|_0^4 = \frac{1}{4} (e^8 - 1) \end{aligned}$$

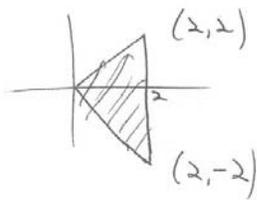
(b) (10 points) Evaluate the double integral  $\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(x^2 + y^2) dy dx$ .

This is over region (quarter of a circle)   $\rightarrow$  integrand involves  $x^2 + y^2$   $\rightarrow$  go polar!

$$\begin{aligned} \text{then limits become } &\int_0^{\pi/2} \int_0^2 \\ \text{get } \int_0^{\pi/2} \int_0^2 \cos(r^2) r dr d\theta &\begin{cases} u = r^2 \\ du = 2r dr \\ \frac{1}{2} du = r dr \end{cases} \begin{cases} \int \cos(r^2) dr \\ = \frac{1}{2} \sin(r^2) \end{cases} \\ = \int_0^{\pi/2} \frac{1}{2} \sin(r^2) \Big|_0^2 d\theta &= \frac{\pi}{2} \left( \frac{1}{2} \sin(4) \right) = \frac{\pi}{4} \sin(4) \end{aligned}$$

**Question 6.** (14 points total)

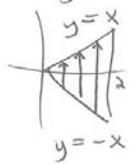
Let  $R$  be the triangular region in the  $xy$ -plane with vertices at the origin and at  $(2, 2)$  and  $(2, -2)$ . Find the volume of the region in 3-space above  $R$  and below the surface given by  $x^2 + z^2 = 4$ .



need  $\iint_R \sqrt{4-x^2} dA$

$z = \sqrt{4-x^2}$   
(above  $xy$  plane)

try  $y$  first to avoid having to split into two regions ...



so get  $\int_0^2 \int_{-x}^x \sqrt{4-x^2} dy dx$

$$= \int_0^2 \left( y \sqrt{4-x^2} \Big|_{y=-x}^x \right) dx = \int_0^2 2x \sqrt{4-x^2} dx$$

$$\left\{ \begin{array}{l} u = 4-x^2 \\ du = -2x dx \\ -du = 2x dx \end{array} \right. \quad \int 2x \sqrt{4-x^2} dx = -\frac{2}{3} (4-x^2)^{3/2}$$

$$\text{so } = -\frac{2}{3} (4-x^2)^{3/2} \Big|_0^2$$

$$= -\frac{2}{3} (0)^{3/2} - \left( -\frac{2}{3} (4)^{3/2} \right) = \frac{16}{3}$$