Math 21a Hourly 2, Spring 97 Solutions

① $\int_0^{\pi/2}\int_0^1\int_0^r (z+r)\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \frac{3}{2} r^3\,dr\,d\theta = \int_0^{\pi/2}\frac{3}{8}\,d\theta = \frac{3\pi}{16}$

② a) $\int_0^4 (2y - y^{3/2})\,dy = y^2 - \frac{2}{5}y^{5/2}\Big|_{y=0}^{4} = \frac{16}{5}$

b) 

c) $\int_0^2\int_0^{(x-2)^2} y\,dy\,dx$

③ a) $\nabla f = 2x\vec{i} + 6y\vec{j} \Rightarrow \nabla f|_Q = 4\vec{i} + 6\vec{j}$

b) $\|\nabla f|_Q\| = \sqrt{16+36} = \sqrt{52}$

c) $\vec{n} = 6\vec{i} - 4\vec{j}$ by switching coordinates and negating one

d) S = level surface $g=0$, where $g = z - x^2 - 3y^2 \Rightarrow \nabla g = -2x\vec{i} - 6y\vec{j} + \vec{k} \Rightarrow \nabla g|_{(2,1,7)} = -4\vec{i} - 6\vec{j} + \vec{k}$

④ Lagrange $\Rightarrow$ $\nabla B = \lambda \nabla A$ and $A = 600$ $\Rightarrow$

$$\left.\begin{aligned} K^{-1/2}L^{1/2} &= \tfrac{1}{2}\lambda(3L+60)K^{-1/2} \\ K^{1/2}L^{-1/2} &= \lambda\, 3K^{1/2} \end{aligned}\right\} \rightarrow L = \frac{1}{2\cdot 3}(3L+60)$$

$(3L+60)K^{1/2} = 600$

So $L = 20$, and thus $K = 25$ from constraint. $B(25,20) = 20\sqrt{5}$.

⑤ a) Substituting $z=-3$ into f gives $f = x^4 + y^2 + 6y + 1 \Rightarrow \begin{aligned} f_x &= 4x^3 = 0 \\ f_y &= 2y+6=0 \end{aligned} \Rightarrow (0,-3)$ is crit. pt.
So critical value is $f = -26$.

b) As $x \to \pm\infty$ or $y \to \pm\infty$, the x^4 and y^2 terms dominate f. So $f \to +\infty$ on the infinite boundaries. This means the critical value in (a) must be a minimum.

⑥ There are no unconstrained crit. pts, since grad f is never $\vec{0}$ (But grad f is undefined when $x=0$ or $y=0$ or $z=0$.) On the $x=0$, $y=0$, or $z=0$ boundaries, $f=0$. On the boundary $g(x,y,z) = 2x+y+z = 4$, use Lagrange. So $\nabla f = \lambda \nabla g \Rightarrow$

$$\left.\begin{aligned} \tfrac{1}{2}x^{-1/2}y^{1/4}z^{1/4} &= 2\lambda \\ \tfrac{1}{4}x^{1/2}y^{-3/4}z^{1/4} &= 1\lambda \\ \tfrac{1}{4}x^{1/2}y^{1/4}z^{-3/4} &= 1\lambda \end{aligned}\right\} \rightarrow y=x,\; z=y$$

Putting $x=y=z$ into $g=4$ gives $4x=4$, or $x=1$.
So $(1,1,1)$ = crit. pt. on boundary, and $f=1$ there is max.

⑦ $\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{3\sqrt{2}} (5\rho^2\cos^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 972\sqrt{2}\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\cos^2\phi\sin\phi\,d\phi\,d\theta = 324\int_0^{2\pi}d\theta$
$= 648\pi.$

⑧ $\int_0^1\int_{-x}^{x}\sqrt{1-x^2}\,dy\,dx = \int_0^1 2x\sqrt{1-x^2}\,dx = 2/3.$

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(9) $\int_{\pi/4}^{\pi/2}\int_1^{\sqrt{2}} r^2\sin^2\theta \, r\cos\theta \, r\,dr\,d\theta = \frac{4\sqrt{2}-1}{5}\int_{\pi/4}^{\pi/2}\sin^2\theta\cos\theta\,d\theta = \frac{4\sqrt{2}-1}{15}\left(1-\frac{\sqrt{2}}{4}\right).$

(10) $\int_0^{2\pi}\int_0^{\pi}\int_1^2 (\rho^2)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{127}{7}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{254}{7}\int_0^{2\pi}d\theta = \frac{508\pi}{7}.$

Volume of R = vol. of rad. 2 sphere − vol. of rad. 1 sphere $= \frac{4}{3}\pi 2^3 - \frac{4}{3}\pi 1^3 = \frac{28\pi}{3}.$

So avg. val. $= \frac{1}{\text{vol.}}\cdot$ integral $= \frac{381}{49}.$