Question 1. (12 points total)

(a) (6 points) Suppose **a** and **b** are two vectors such that **b** is twice as long as **a**, and the angle between **a** and **b** is 60 degrees. Calculate the value of $\dfrac{(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})}{|\mathbf{a}|^2}$

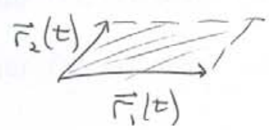
so $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos 60^\circ = |\vec{a}|(2|\vec{a}|)\frac{1}{2} = |\vec{a}|^2 = \vec{a}\cdot\vec{a}$

then $(\vec{a}+2\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{a} - 2\vec{b}\cdot\vec{b}$

$= |\vec{a}|^2 + \vec{a}\cdot\vec{b} - 2|2\vec{a}|\,|2\vec{a}| = |\vec{a}|^2 + |\vec{a}|^2 - 8|\vec{a}|^2$

so altogether $\dfrac{(\vec{a}+2\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}|^2} = \dfrac{-6|\vec{a}|^2}{|\vec{a}|^2} = -6$

(b) (6 points) Let $\mathbf{r}_1(t) = \langle t, \ 2, \ 1\rangle$ and $\mathbf{r}_2(t) = \langle t, \ 1, \ -2\rangle$ be two vector valued functions, where t represents time in seconds. Consider the parallelogram determined by vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. What is the area of this parallelogram when $t = 1$? What is the rate at which the area of the parallelogram is changing at this time?



area $= |\vec{r}_1(t) \times \vec{r}_2(t)|$

$$\begin{vmatrix} i & j & k \\ t & 2 & 1 \\ t & 1 & -2 \end{vmatrix} = \langle -5, 3t, -t\rangle$$

magnitude $= \sqrt{25 + 9t^2 + t^2} = \sqrt{25+10t^2}$

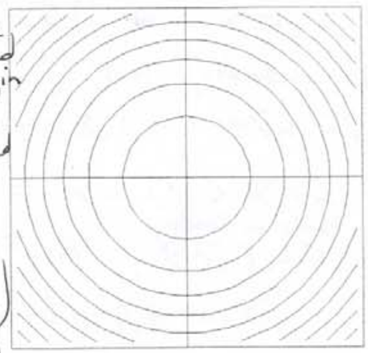
So area $= \sqrt{25+10} = \sqrt{35}$ when $t = 1$

rate of change with respect to t ... take derivative of $\sqrt{25+10t^2}$ ... $\frac{1}{2}(25+10t^2)^{-1/2} \times (20t) = \dfrac{10t}{\sqrt{25+10t^2}}$

when $t=1 \Rightarrow \dfrac{10}{\sqrt{35}}$ (square units per second)

Question 2 (9 points total) Circle the correct answer for each of the following. Assume that the level curves have been plotted for evenly spaced values of $f(x,y)$. For each set of level curves, chose the function of two variables that is depicted by circling the (one) correct function.

(a) (3 points)



(a) $f(x,y) = \dfrac{1}{x^2+y^2}$ $\longrightarrow$ circles more closely spaced towards origin

(b) $f(x,y) = \sqrt{x^2+y^2}$ $\longrightarrow$ evenly spaced circles

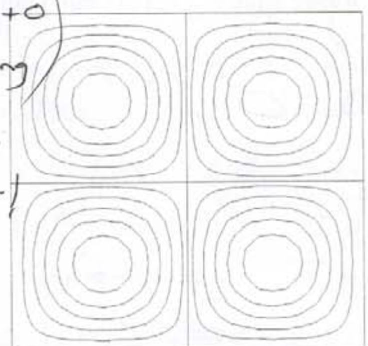
(c) $f(x,y) = x^2+y^2$ $\longrightarrow$ this is it!

(d) $f(x,y) = x+y$ $\longrightarrow$ (parallel lines not circles)

(e) $f(x,y) = \dfrac{1}{x+y}$ $\longrightarrow$ (parallel lines not circles)

(b) (3 points)

(when $y=0$, $\sin(x) + 0$ would change value (i.e. change levels) along x axis)



(a) $f(x,y) = \sin(x) + \sin(y)$

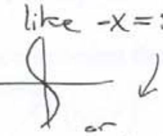
(b) $f(x,y) = \sin(x+y)$ $\longrightarrow$ parallel lines

(c) $f(x,y) = \sin(x)\sin(y)$ $\longrightarrow$ This is it!

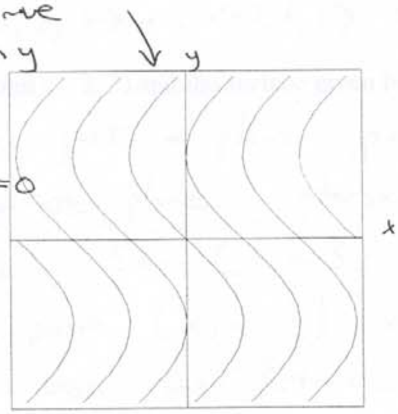
(d) $f(x,y) = \sin(x^2+y^2)$ $\longrightarrow$ circles

(e) $f(x,y) = \sin(\dfrac{1}{x+y})$ $\longrightarrow$ parallel lines

note one level curve looks like $-x = \sin y$



(c) (3 points)



or $x + \sin(y) = 0$ aha!

(a) $f(x,y) = x + \cos(y)$

(b) $f(x,y) = x - \cos(y)$

(c) $f(x,y) = \sin(x) - \cos(y)$

(d) $f(x,y) = x + \sin(y)$

(e) $f(x,y) = \sin(x+y)$

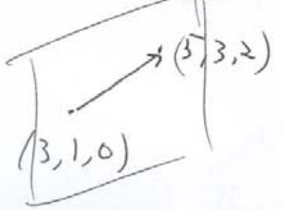
(a) and (b) would have level curves that are tangent to a coordinate axes at the origin unlike the diagram's level curves

Question 3. (16 points total)

(a) (6 points) Find the equation of a plane that passes through the two points (5, 3, 2) and (3, 1, 0) and that never intersects the y-axis.

Never intersects y-axis ⟶ aha, must be parallel to y-axis, i.e. parallel to $\langle 0,1,0\rangle$. Another vector parallel (or "in") the plane is the difference

$\langle 5,3,2\rangle - \langle 3,1,0\rangle = \langle 2,2,2\rangle$



So normal $= \langle 0,1,0\rangle \times \langle 2,2,2\rangle = \langle 2,0,-2\rangle$

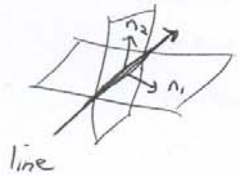
then get $\vec{r}\cdot\vec{n} = \underbrace{\langle 5,3,2\rangle}_{\text{(point on plane)}} \cdot \overbrace{\langle 2,0,-2\rangle}^{\vec{n}} = 6$

$= 2x + 0y - 2z = 6$ or $x - z = 3$

(b) (6 points) Consider two planes with normal vectors $\mathbf{n}_1 = \langle 20,40,0\rangle$ and $\mathbf{n}_2 = \langle 0,26,-26\rangle$, respectively. Suppose both planes contain the point (1, 1, 1). Find a vector function that parametrizes the line of intersection of these two planes.

yuck.. try normals $\langle 1,2,0\rangle$ and $\langle 0,1,-1\rangle$
(magnitudes don't matter, just directions).

pretty simple then ⟶ line will be along (parallel to) vector that's perpendicular to both normals... i.e. direction vector for line is given by $\vec{n}_1 \times \vec{n}_2$, or

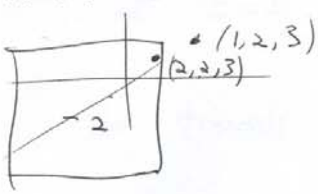


$\langle 1,2,0\rangle \times \langle 0,1,-1\rangle = \langle -2,1,1\rangle$, and

contains $(1,1,1) \Longrightarrow$

$\vec{r}(t) = \langle 1,1,1\rangle + t\langle -2,1,1\rangle = \langle 1-2t, 1+t, 1+t\rangle$

(c) (4 points) Find the distance between the point (1, 2, 3) and the surface given by $\mathbf{r}(u,v) = \langle 2,u,v\rangle$



$\vec{r}(u,v)$ is just a plane parallel to yz-coordinate plane intersecting x-axis at $x=2$. So $(1,2,3)$ is closest to the point $(2,2,3)$ on this plane in which case the distance $= 1$

(can check $(2,2,3)$ is closest as $\langle 2,2,3\rangle - \langle 1,2,3\rangle = \langle 1,0,0\rangle$ is perpendicular to the plane)

Question 4. (12 points total)

(a) (6 points) At spring training a pitcher on the Math 21a baseball team gets ready to pitch a baseball. During the pitch the pitcher's hand (the one holding the ball) traces a curve given by the parametric equations $x = \cos(t)$, $y = \sin(t)$, and $z = 6 - 2t$. The pitcher begins the pitch at $t = 0$ and lets go of the ball when $t = \frac{\pi}{2}$. How far does the ball in the pitcher's hand travel along this curve during this time?

if $\vec{r}(t) = \langle \cos t, \sin t, 6-2t \rangle$ then

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, -2 \rangle$$

and $|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$

then arc length for $0 \le t \le \frac{\pi}{2}$

is $\displaystyle\int_0^{\pi/2} |\vec{r}\,'(t)|\,dt = \int_0^{\pi/2} \sqrt{5}\,dt = \frac{\sqrt{5}\,\pi}{2}$

(b) (6 points) Given that the pitcher releases the ball when $t = \frac{\pi}{2}$, where is the ball 2 seconds later? (Keep it simple in the sense of neglecting gravity and air resistance!)

now ball continues in straight line in direction of $\vec{r}\,'(t)$ when $t = \frac{\pi}{2}$, i.e. in direction $\vec{r}\,'(\frac{\pi}{2}) = \langle -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, -2 \rangle$

$$= \langle -1, 0, -2 \rangle$$

at $t = \frac{\pi}{2}$ seconds ball is at $\vec{r}(\frac{\pi}{2}) = \langle \cos\frac{\pi}{2}, \sin\frac{\pi}{2}, 6 - 2(\frac{\pi}{2}) \rangle$

$$= \langle 0, 1, 6-\pi \rangle$$

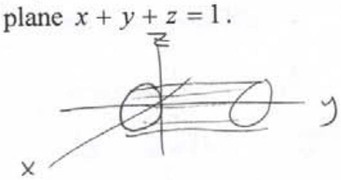
so travels to $\vec{r}(\frac{\pi}{2}) + 2 \times \vec{r}\,'(\frac{\pi}{2})$

$$= \langle 0, 1, 6-\pi \rangle + 2\langle -1, 0, -2 \rangle$$

$$= \langle 0-2, 1, 6-\pi-4 \rangle = (-2, 1, 2-\pi)$$

Question 5. (15 points total)

(a) (7 points) Find a parametrization for the curve of intersection of the cylinder $(x-1)^2 + z^2 = 1$ and the plane $x + y + z = 1$.



if had $x^2 + z^2 = 1$ would just use

$\langle \cos\theta, \dots, \sin\theta \rangle$ for x, z.

x is shifted one to positive x direction

so need $\langle \cos\theta + 1, \dots, \sin\theta \rangle$

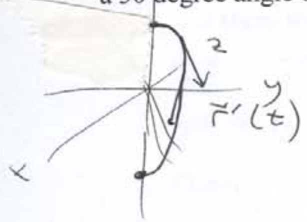
Now from $x+y+z=1$ we know $y = 1 - x - z$,

so $y = 1 - (\cos\theta + 1) - \sin\theta = -\cos\theta - \sin\theta$.

So in total $\vec{r}(\theta) = \langle \cos\theta + 1, -\cos\theta - \sin\theta, \sin\theta \rangle$

should do it for the curve of intersection's parametrization.

(b) (8 points) Suppose a somewhat strange Math 21a rollercoaster ride follows a path described by the vector function $\mathbf{r}(t) = \left\langle -\sqrt{2}\sin(t),\ -\sqrt{2}\sin(t),\ 2\cos(t) \right\rangle$, where t is measured in seconds. The ride lasts for π seconds (i.e. $0 \le t \le \pi$). At what times during the ride is the rollercoaster either going up at a 30 degree angle or down at a 30 degree angle (where the z-axis is considered to be vertical)?



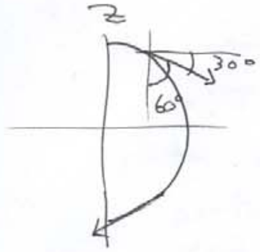
semicircle path from $(0,0,2)$ to $(0,0,-2)$

Now find places where $\vec{r}'(t)$

$= \langle -\sqrt{2}\cos t, -\sqrt{2}\cos t, -2\sin t \rangle$ makes

a 30° angle with xy plane, or

better yet, a 60° (or 120°) angle with the (positive) z axis.



so $r'(t) \cdot \langle 0,0,1 \rangle = |\vec{r}'(t)|\ |\langle 0,0,1 \rangle| \cos 60^\circ$

or $-2\sin t = \sqrt{2\cos^2 t + 2\cos^2 t + 4\sin^2 t}\ \cos 60^\circ$

$= 2 \cdot \frac{1}{2}$ or $2 \cdot \left(-\frac{1}{2}\right)$

so $\sin t = \pm \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ seconds.

(within the $0 \le t \le \pi$ bounds)

Question 6. (16 points total)

(a) (4 points) Can there be any vectors **v** such that $\mathbf{v} \times \langle 2,3,1\rangle = \langle 1,1,1\rangle$? Find a solution vector or explain why this is not possible.

No, it's impossible ... whatever $\vec{v} \times \langle 2,3,1\rangle$ equals, it will be a vector that is perpendicular to both $\vec{v}$ and to $\langle 2,3,1\rangle$, and so this can't equal $\langle 1,1,1\rangle$, as $\langle 1,1,1\rangle$ isn't $\perp$ to $\langle 2,3,1\rangle$

check $\langle 2,3,1\rangle \cdot \langle 1,1,1\rangle \neq 0$

(b) (4 points) Suppose **v** and **w** are vectors such that $\mathbf{v} \times \mathbf{w} = \langle 1,1,1\rangle$. Explain why it is that all the pairs of vectors that satisfy this equation lie on one specific plane through the origin. Find that particular plane.

Any pair $\vec{v}, \vec{w}$ with $\vec{v} \times \vec{w} = \langle 1,1,1\rangle$ must be vectors that themselves are perpendicular to $\langle 1,1,1\rangle$. Aha $\rightarrow$ $\langle 1,1,1\rangle$ is a normal for the plane in question. If $(0,0,0)$ is on the plane as well then the equation is simply just $x+y+z=0$, (and since $\vec{v}, \vec{w}$ are $\perp$ to $\langle 1,1,1\rangle$, then they "lie in" or are parallel to this plane)

(c) (8 points) Let two space curves, R and S be described by vector functions $\mathbf{r}(t)$ and $\mathbf{s}(t)$, respectively. Suppose that $\mathbf{r}(t)$ is parallel to $\mathbf{s}'(t)$, and that in addition $\mathbf{s}(t)$ is parallel to $\mathbf{r}'(t)$. By considering the derivative of the cross product of $\mathbf{r}(t)$ and $\mathbf{s}(t)$, show that there must be a particular plane P through the origin such that the two space curves R and S lie on P (you don't have enough information to actually find P, just show that there must be such a plane).

If $\vec{r}(t)$ is parallel to $\vec{s}'(t)$ then $\vec{r}(t) \times \vec{s}'(t) = \vec{0}$

similarly $\vec{r}'(t) \times \vec{s}(t) = \vec{0}$

Then $\frac{d}{dt}\left(\vec{r}(t) \times \vec{s}(t)\right) = \vec{r}(t) \times \vec{s}'(t) + \vec{r}'(t) \times \vec{s}(t) = \vec{0}$

i.e. $\vec{r}(t) \times \vec{s}(t) =$ constant vector. So $\vec{r}(t) \times \vec{s}(t) = \vec{v}$ some $\vec{v}$, which means $\vec{r}(t)$ and $\vec{s}(t)$ are perpendicular to $\vec{v}$ for all t, so $\vec{r}(t)$ and $\vec{s}(t)$ are parallel to all planes with $\vec{v}$ as their normal, in particular the plane P through the origin. Since $\vec{r}(t)$ and $\vec{s}(t)$ are position vectors for R and S, then R and S lie on P.

Question 7. (20 points total)

(a) (6 points) Consider the following five regions in 3-space:

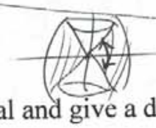
(i) $\rho \le 1$ and $0 \le \varphi \le \pi/4$ $\qquad \rho \le 1$ 

$\neq (\rho \le 1)$

(ii) $z \le 1$ and $r^2 \le z^2$ 

(iii) $r^2 + z^2 \le 1$ and $r^2 \ge z^2$ $\qquad \rho \le 1 \qquad r^2 \geq z^2$

(iv) $x^2 + y^2 + z^2 \le 1$ and $x^2 + y^2 \le z^2$ $\qquad \rho \le 1 \qquad r^2 \le z^2$ 

(v) $\rho \le 1$ and $\pi/4 \le \varphi \le 3\pi/4$ $\qquad \rho \le 1$ 

Exactly two of these regions are identical. Identify which two are identical and give a detailed explanation as to why you know it's the two you've identified.

From sketches it appears that 3 and 5 are identical. Check:

① $r^2 + z^2 \le 1$ is equivalent to $(x^2+y^2)+z^2 \le 1$

or $\rho \le 1$

② note $r = \rho \sin\varphi$ and $z = \rho\cos\varphi$

(know this or note $= \sqrt{x^2+y^2} = \sqrt{(\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2} = \rho\sin\varphi$)

So $r^2 \ge z^2$ is equivalent to $\rho^2\sin^2\varphi \ge \rho^2\cos^2\varphi$

so $\sin^2\phi \ge \cos^2\phi$ or $|\sin\phi| \ge |\cos\phi|$

in the range $0 \le \phi \le \pi$ this will hold

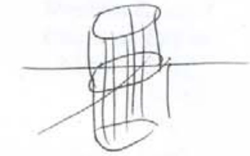
whenever $\pi/4 \le \varphi \le 3\pi/4$

Question 7 continued

(b) (9 points) Consider the surface parametrized by $\mathbf{r}(u, v) = \langle \cos(v), \ \sin(v), \ u \rangle$. Write equations describing this surface in rectangular, cylindrical and spherical coordinates.

sketch: cos, sin in x, y $\Rightarrow$ circular cross sections

u in z coordinate $\Rightarrow$ slide circular cross section up / down z-axis



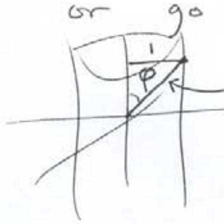
$\Rightarrow$ cylinder

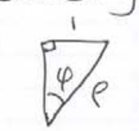
aha... rectangular $x^2 + y^2 = 1$ (z can be anything)

so in cylindrical $r^2 = 1$ or... $r = 1$

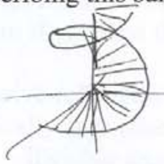
in spherical hmm... could simply note $r = \rho \sin\varphi$ so $\rho \sin\varphi = 1$,

or go from geometry:



so  $\Rightarrow$ $\sin\varphi = \frac{1}{\rho}$ i.e. $\rho \sin\varphi = 1$

(c) (5 points) Consider the surface parametrized by $\mathbf{r}(u, v) = \langle u\cos(v), \ u\sin(v), \ v \rangle$. Find an equation describing this surface in cylindrical coordinates.



looks like a spiral staircase

note if you consider $u = r$, $v = \theta$

then $x = r\cos\theta = x$ $\quad y = r\sin\theta = y$

then $z = v = \theta$

i.e. $x = x$, $y = y$, $z = \theta$, or more simply $\boxed{z = \theta}$ is the equation that generates this in cylindrical coordinates. Check if given $z = \theta$, then translate back to rectangular coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, i.e.

$\langle r\cos\theta, r\sin\theta, \theta \rangle \Rightarrow$ equivalent to $\vec{r}(u, v) = \langle u\cos v, u\sin v, v \rangle$