**Math 21a Fall 2004** **Course Head: Andy Engelward**

# Solutions: HW Section 11.3

#6, 8, 18, 23, 30, 51, 60

**6.** (a) The graph of $f$ decreases if we start at $(-1, 2)$ and move in the positive $x$-direction, so $f_x(-1, 2)$ is negative.

(b) The graph of $f$ decreases if we start at $(-1, 2)$ and move in the positive $y$-direction, so $f_y(-1, 2)$ is negative.

(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so $f_{xx}$ is the rate of change of $f_x$ in the $x$-direction. $f_x$ is negative at $(-1, 2)$ and if we move in the positive $x$-direction, the surface becomes less steep. Thus the values of $f_x$ are increasing and $f_{xx}(-1, 2)$ is positive.

(d) $f_{yy}$ is the rate of change of $f_y$ in the $y$-direction. $f_y$ is negative at $(-1, 2)$ and if we move in the positive $y$-direction, the surface becomes steeper. Thus the values of $f_y$ are decreasing, and $f_{yy}(-1, 2)$ is negative.

**8.** $f_x(2, 1)$ is the rate of change of $f$ at $(2, 1)$ in the $x$-direction. If we start at $(2, 1)$, where $f(2, 1) = 10$, and move in the positive $x$-direction, we reach the next contour line (where $f(x, y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2, 1)$ from the left (moving in the positive $x$-direction) the output values increase from 8 to 10 with an increase in $x$ of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2, 1)$ would be the average of these two, so $f_x(2, 1) \approx 2.8$. Similarly, $f_y(2, 1)$ is the rate of change of $f$ at $(2, 1)$ in the $y$-direction. If we approach $(2, 1)$ from below, the output values decrease from 12 to 10 with a change in $y$ of approximately 1 unit, corresponding to an average rate of change of $-2$. If we start at $(2, 1)$ and move in the positive $y$-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2, 1) \approx -2.1$.

**18.** $f(x, y) = x^y \quad \Rightarrow \quad f_x(x, y) = yx^{y-1}$, $f_y(x, y) = x^y \ln x$

**23.** $z = \ln\left(x + \sqrt{x^2 + y^2}\right) \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{x + \sqrt{x^2 + y^2}}\left[1 + \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x)\right] = \frac{\left(\sqrt{x^2 + y^2} + x\right)\Big/\sqrt{x^2 + y^2}}{\left(x + \sqrt{x^2 + y^2}\right)} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x + \sqrt{x^2 + y^2}}\left(\frac{1}{2}\right)(x^2 + y^2)^{-1/2}(2y) = \frac{y}{x\sqrt{x^2 + y^2} + x^2 + y^2}$$

**30.** $u = x^{y/z} \quad \Rightarrow \quad u_x = \dfrac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z} \ln x \cdot \dfrac{1}{z} = \dfrac{x^{y/z}}{z} \ln x$, $u_z = x^{y/z} \ln x \cdot \dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2} \ln x$

**51.** $u = \ln\sqrt{x^2+y^2} = \ln\left(x^2+y^2\right)^{1/2} = \frac{1}{2}\ln\left(x^2+y^2\right) \quad\Rightarrow\quad u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)\left(x^2+y^2\right)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)\left(x^2+y^2\right)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

**60.** (a) If we fix $y$ and allow $x$ to vary, the level curves indicate that the value of $f$ decreases as we move through $P$ in the positive $x$-direction, so $f_x$ is negative at $P$.

(b) If we fix $x$ and allow $y$ to vary, the level curves indicate that the value of $f$ increases as we move through $P$ in the positive $y$-direction, so $f_y$ is positive at $P$.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix $y$ and allow $x$ to vary, $f_{xx}$ is the rate of change of $f_x$ as $x$ increases. Note that at points to the right of $P$ the level curves are spaced farther apart (in the $x$-direction) than at points to the left of $P$, demonstrating that $f$ decreases less quickly with respect to $x$ to the right of $P$. So as we move through $P$ in the positive $x$-direction the (negative) value of $f_x$ increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at $P$.