Name:

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| TTH 115 Thomas Lam |

- Please mark the box to the left which lists your section.

- Do not detach pages from this exam packet or unstaple the packet.

- Show your work. Answers without reasoning can not be given credit except for the True/False and multiple choice problems.

- Please write neatly.

- Do not use notes, books, calculators, computers, or other electronic aids.

- Unspecified functions are assumed to be smooth and defined everywhere unless stated otherwise.

- You have 180 minutes time to complete your work.

- The Biochem section can ignore problems with vector fields and line integrals.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10

12A		10
13A		10
14A		10

12B		10
13B		10
14B		10

$$\boxed{\text{Total:} \quad | \quad | \quad 140}$$

Problem 1) True/False questions (20 points)

1) [T] [**F**]   For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v}$ is parallel to $\vec{w}$.

Solution:
Take $\vec{v} = \langle 1, 0, 0 \rangle, \vec{w} = \langle 0, 1, 0 \rangle$ so that $\vec{v} \times \vec{w} = \langle 0, 0, 1 \rangle$ and $(\vec{v} \times \vec{w}) \times \vec{v} = \langle 0, 1, 0 \rangle$ and $((\vec{v} \times \vec{w}) \times \vec{v}) \times \vec{v} = \langle 0, 0, 1 \rangle$.

2) [T] [**F**]   The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

Solution:
Take $\vec{v} = \vec{w}$, then the right hand side is the zero vector while the left hand side is not zero in general (for example if $\vec{u} = \vec{i}, \vec{v} = \vec{j}$).

3) [**T**] [F]   If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

Solution:
One can see that with the formula $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$ which shows that the acceleration $\vec{r}\,''(t)$ is in the velocity direction at all times. One can also see it intuitively or with the definition $\kappa(t) = \vec{T}\,'(t)/|\vec{T}\,'(t)|$. If curve is not part of a line, then $\vec{T}\,'$ has to change which means that κ is not zero somewhere.

4) [**T**] [F]   The curve $\vec{r}(t) = (1-t)A + tB, t \in [0,1]$ connects the point A with the point B.

Solution:
The curve is a parameterization of a line and for $t = 0$, one has $\vec{r}(0) = A$ and for $t = 1$ one has $\vec{r}(1) = B$.

5) [**T**] [F]   For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

Solution:
Just differentiate.

6) $\boxed{\text{T}} \boxed{\text{F}}$ The length of the curve $\vec{r}(t) = (t, \sin(t))$, where $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1 + \cos^2(t)}\, dt$.

Solution:
The speed at time t is $|\vec{r}'(t)| = \sqrt{1 + \cos^2(t)}$.

7) $\boxed{\text{T}} \boxed{\text{F}}$ Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

Solution:
While this would be true for $g(x, y) = f(y)$, where the constraint is a straight line parallel to the y axis, it is false in general.

8) $\boxed{\text{T}} \boxed{\text{F}}$ The function $f(x, y, z) = x^2 - y^2 - z^2$ decreases in the direction $(2, -2, -2)/\sqrt{8}$ at the point $(1, 1, 1)$.

Solution:
It **increases** in that direction.

9) $\boxed{\text{T}} \boxed{\text{F}}$ Assume $\vec{F}$ is a vector field satisfying $|\vec{F}(x, y, z)| \leq 1$ everywhere. For every curve $C : \vec{r}(t)$ with $t \in [0, 1]$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is less or equal than the arc length of C.

Solution:
$|\vec{F} \cdot \vec{r}'| \leq |\vec{F}||\vec{r}'| \leq |\vec{r}'|$

10) $\boxed{\text{T}} \boxed{\text{F}}$ Let $\vec{F}$ be a vector field which coincides with the unit normal vector $\vec{N}$ for each point on a curve C. Then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:
The vector field is orthogonal to the tangent vector to the curve.

11) $\boxed{\text{T}} \boxed{\text{F}}$ If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a, b, c)$, where a, b, c are constants.

Solution:
One can also have $\vec{F} = \vec{G} + \text{grad}(f)$ which are vectorfields with the same curl.

12) **T** F   If a nonempty quadric surface $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.

Solution:
If one or two of the constants a, b, c are negative, we have a hyperboloid which all can not be contained into a finite space. If all three are negative, then the surface is empty.

13) T **F**   If $\mathrm{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z), then $\mathrm{curl}(\vec{F}) = (0, 0, 0)$ for all (x, y, z).

Solution:
There are counter examples: take $(-y, x, 0)$ for example.

14) **T** F   If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

Solution:
Otherwise, it is would be a cone (or for $\alpha = 0$ or $\alpha = \pi$ a half line).

TF PROBLEMS FOR REGULAR AND PHYSICS SECTIONS:

15) T **F**   The divergence of the gradient of any $f(x, y, z)$ is always zero.

Solution:
$\mathrm{div}(\mathrm{grad}(f)) = \Delta f$ is the Laplacian of f.

16) T **F**   For every vector field $\vec{F}$ the identity $\mathrm{grad}(\mathrm{div}(\vec{F})) = \vec{0}$ holds.

Solution:
$F = (x^2, y^2, z^2)$ has $\mathrm{div}(F) = (2x, 2y, 2z)$ which has a nonzero gradient.

17) **T** F   For every function f, one has $\mathrm{div}(\mathrm{curl}(\mathrm{grad}(f))) = 0$.

Solution:
Both because $\text{div}(\text{curl}(F)) = 0$ and $\text{curl}(\text{grad}(f)) = 0$.

18) | T || **F** | If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

Solution:
While it is true that the flux of $\text{curl}(F)$ vanishes through every closed surface, this is not true for $\vec{F}$ itself. Take for example $F = (x, y, z)$.

19) | T || **F** | The flux of the vector field $\vec{F}(x, y, z) = (y + z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

Solution:
By the divergence theorem, the flux through the boundary is $\int \int \int_E \text{div}(F) \ dV$ but $\text{div}(F) = 0$. So the flux is zero.

20) | **T** || F | For every function $f(x, y, z)$, there exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = f$.

Solution:
In order to solve $P_x + Q_y + R_z = f$ just take $F = (0, 0, \int_0^z f(x, y, w) \ dw)$.

TF PROBLEMS FOR BIOCHEM SECTIONS:

21) | T || **F** | The following reasoning is correct: tossing 3 unbiased coins, the possible numbers of heads appearing are $0, 1, 2$, and 3. Therefore each of these events has probability $1/4$.

Solution:
There are 8 possibilities $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$. The probability to have no head is $1/8$, the probability to have 1 head is $3/8$. The probability to have 2 head is $3/8$. The probability to have 3 head is $1/8$.

22) | **T** || F | Two events A, B for which $P(B) > 0$ are independent if and only if $P(A|B) = P(A)$

Solution:
By definition, $P(A|B) = P(A \cap B)/P(B) = P(A)P(B)/P(B) = P(A)$

23) T **F**
For two independent random variables X, Y one has the following identities for the variance $D(X) - D(Y) = D(X - Y)$.

Solution:
Take $Z = -Y$. Then the identity under investigation is $D(X) - D(Z) = D(X + Z)$ because $D(-Y) = D(Y)$. But the correct identity is $D(X) + D(Z) = D(X + Z)$.

24) T **F**
Let A, B be arbitrary events. If $P(A|B) = P(B|A)$ then $P(A) = P(B)$.

Solution:
$P(A|B) = P(A \cap B)/P(B)$ and $P(B|A) = P(A \cap B)/P(A)$. With equality, we have $P(A) = P(B)$. There is however a catch: we could have the case $P(A \cap B) = 0$ in which case still $P(A|B) = P(B|A)$ but with no relation between P(A) and P(B).

25) **T** F
The probability that from 6 random coins tossed, 6 show tail is smaller than the probability that 5 show tail. Each coin has probability 1/2 to show tail.

Solution:
The probability that 5 show tail is $Bin(6,5)/2^6 = 6/64$. The probability that all shows tail is $1/2^6$.

26) T **F**
If you throw 2 dice and you know the first one shows the number 1, then the chance that the second one shows 1 is less than 1/6.
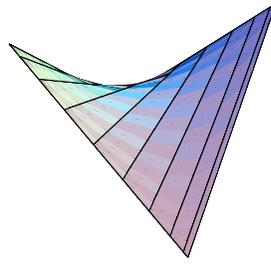
Solution:
The events are independent. The two events A, B satisfy $P(A|B) = P(A)$.
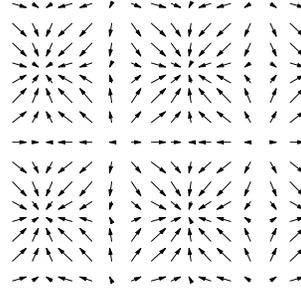

Problem 2) (10 points)

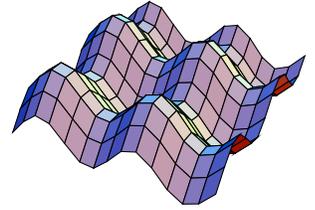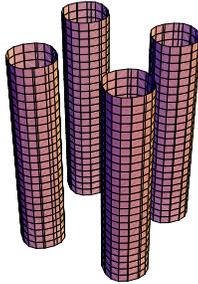Match the equations with the objects. No justifications are needed.
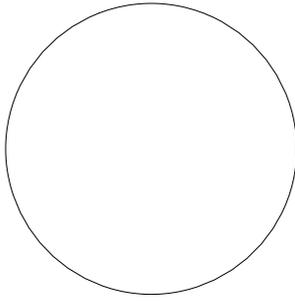
I II III IV
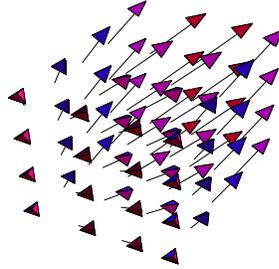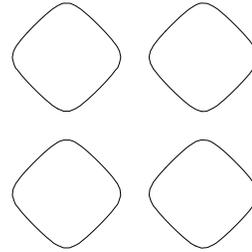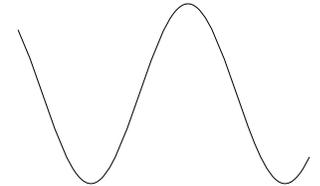
V VI VII VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x,y,z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = (\cos(t), \sin(t))$
	$\vec{r}(u,v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
	$\vec{F}(x,y,z) = (\cos(x), \sin(x), 1)$
	$z = f(x,y) = \cos(x) + \sin(y)$
	$g(x,y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x,y) = (\cos(x), \sin(x))$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
V	$g(x, y, z) = \cos(x) + \sin(y) = 1$
VIII	$y = \cos(x) - \sin(x)$
I	$\vec{r}(t) = (\cos(t), \sin(t))$
II	$\vec{r}(u, v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
VI	$\vec{F}(x, y, z) = (\cos(x), \sin(x), 1)$
IV	$z = f(x, y) = \cos(x) + \sin(y)$
VII	$g(x, y) = \cos(x) - \sin(y) = 1$
III	$\vec{F}(x, y) = (\cos(x), \sin(x))$

Problem 3) (10 points)

Mark with a cross in the column below "conservative" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x, y, z) = (0, 0, 0)$ for all points (x, y, z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z)). No justifications are needed.

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$		
$\vec{F}(x, y, z) = (x, y, z)$		
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$		

Solution:

Vectorfield	conservative $\mathrm{curl}(\vec{F}) = \vec{0}$	incompressible $\mathrm{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$	X	X
$\vec{F}(x, y, z) = (x, y, z)$	X	
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$	X	

Problem 4) (10 points)

Let E be a parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

a) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

b) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

c) (4 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Solution:
a) first diagonal $\boxed{u + v}$, second diagonal $\boxed{u - v}$.
b) $(u + v) \times (u - v) = 2v \times u = \boxed{2 \text{ times area of of parallelogram}}$.
c) $(u + v) \cdot (u - v) = |u|^2 - |v|^2 = 0$, so that $\boxed{|u| = |v|}$.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Solution:
The volume of the box is $8xyz$. The Lagrange equations are

$$8yz = \lambda x/2$$
$$8xz = \lambda 2y/9$$
$$8xy = \lambda 2z/25$$
$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1 = 0$$

We can solve this by solving the first three equations for λ and expressing y, z by x, plugging this into the fourth equation. An other way to solve this is to multiply the first equation with x, the second with y and third with z.

The solution is $\boxed{x = 2/\sqrt{3}, y = \sqrt{3}, z = 5/\sqrt{3}}$. The maximal volume is $8xyz = 80/\sqrt{3}$.
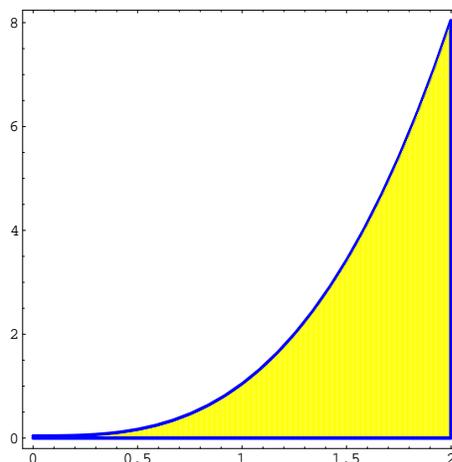

Problem 6) (10 points)

Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} \, dx dy.$$

Solution:
This type II integral can not be computed as it is. We write it as a type I integral: from the boundary relation $x = y^{1/3}$ we obtain $y = x^3$ and $y = 8$ corresponds to $x = 2$:

$$\int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} \, dy dx$$
$$\int_0^2 \frac{x^9}{3} \frac{e^{x^2}}{x^8} \, dx$$
$$\int_0^2 \frac{x}{3} e^{x^2} \, dx$$
$$e^{x^2}/6 \big|_0^2 = (e^4 - 1)/6$$

The result is $\boxed{\frac{e^4-1}{6}}$.

Problem 7) (10 points)

In this problem we evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} \, dxdy$, where D is the triangular region bounded by the x and y axis and the line $x + y = 1$.

a) (3 points) Find the region R in the uv-plane which is transformed into D by the change of variables $u = x - y, v = x + y$. (It is enough to draw a carefully labeled picture of R.)

b) (3 points) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $(x, y) = (\frac{u+v}{2}, \frac{v-u}{2})$.

c) (4 points) Evaluate $\int \int_D \frac{(x-y)^4}{(x+y)^4} \, dxdy$ using the above defined change of variables.

Hint. The general topic of change of variables does not appear this semester. You can solve the problem nevertheless, when given the formula $\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ for the integration factor (analoguous to r when changing to polar coordinates, or $\rho^2 \sin(\phi)$ when going to spherical coordinates). The integral in c) becomes then $\int \int_R u^4/v^4 \, dudv$. The region R is the triangle bounded by the edges $(0, 0), (1, 1), (-1, 1)$.
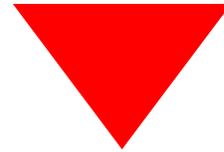
Solution:
a) Take R with the edges $(0,0), (1, 1)$ and $(-1, 1)$.
b) The Jacobian is $x_u y_v - x_v y_u = (1/4 + 1/4) = 1/2$.
c) It is best evaluated as a type II integral:
$\frac{1}{2} \int_0^1 \int_{-v}^{v} u^4/v^4 \, dudv = \frac{1}{2} \int_0^1 2v/5 = \, dv = \boxed{1/10}$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$.
b) (3 points) Classify the critical points.
c) (2 points) Locate the local and absolute maxima of f.
d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

Solution:
a) $(\pm2, 0)$ and $(0, 0)$.

b) $(-2, 0)$ is a local maximum with value 15.

$(0, 0)$ is a saddle with value -1.

$(2, 0)$ is a maximum with value 15.

c) The local maxima are $(\pm2, 0)$. They are also the absolute maxima because f decays at infinity.

d) To calculate the tangent plane at the maximum, write the graph of f as a level surface $g(x, y, z) = z - f(x, y)$. The gradient of g is orthogonal to the surface. We have $\nabla g = (0, 0, 1)$ so that the tangent plane has the equation $z = d = const$. Plugging in the point $(\pm2, 0, 15)$ shows that $z = 15$ is the equation for the tangent plane for both maxima.

Problem 9) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$.

Solution:
Use polar coordinates and note that the wedge is above the right side of the unit disc:

$$\int_0^2 \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta) \, d\theta dr = 16/3$$

The solution is $16/3$.
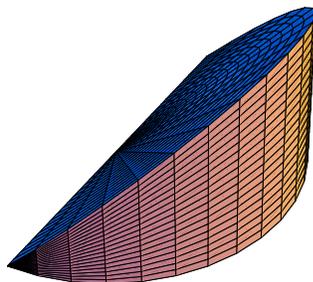
Problem 10) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = (t, \sin t, t^2 \cos t)$ for $0 \le t \le \pi$. Let $f(x, y, z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals. The result is $f(r(\pi)) - f(r(0)) = f(\pi, 0, -\pi^2) - f(0, 0, 0) = \pi^4 e^\pi + \pi^2 - 0 = \boxed{\pi^4 e^\pi + \pi^2}$.

Problem 11) (10 points)

A cylindrical building $x^2 + (y - 1)^2 = 1$ is intersected with the paraboloid $z = 4 - x^2 - y^2$.

a) Parametrize the intersection curve and set up an integral for its arc length.

b) Find a parametrization of the surface obtained by intersecting the paraboloid with the solid cylinder $x^2 + (y - 1)^2 \le 4$ and set up an integral for its surface area.

Solution:
a) $\vec{r}(t) = \langle \cos(t), \sin(t) + 1, 4 - \cos^2(t) - (1 + \sin(t))^2 \rangle$.
Write down $\int_0^{2\pi} |r'(t)| \, dt = \int_0^{2\pi} \sqrt{3 + 2\cos(2t)} \, dt$.
b) $\vec{r}(u, v) = \langle u, v, 4 - u^2 - v^2 \rangle$.
So, the integral is

$$\int\int_{u^2+(v-1)^2 \le 1} \sqrt{1 + 4u^2 + 4v^2} \, du\, dv = \int_{-2}^{2} \int_{-\sqrt{4-u^2}+1}^{\sqrt{4-u^2}+1} \sqrt{1 + 4u^2 + 4v^2} \, dv\, du$$

SECTION SPECIFIC PROBLEMS: PROBLEMS FOR REGULAR AND PHYSICS SEC-
TIONS:

Problem 12A) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x, y) = (y^2, x^2)$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0,0), (1,0)$ and $(1,1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Greens theorem.

Solution:
The problem asks to do this in the clockwise direction. We do it in the counterclockwise direction and change then the sign.
a) $\int_0^1 F(t,0)\cdot(1,0)\ dt + \int_0^1 F(1,t)\cdot(0,1)\ dt + \int_0^1 F(1-t, 1-t)\cdot(-1,-1)\ dt = 0+1-2/3 = 1/3$.
So, the result for the clockwise direction is $\boxed{-1/3}$.
b) The curl of F is $2x - 2y$.

$$\int_0^1 \int_0^x (2x - 2y) dy\ dx = \int_0^1 2x^2 - x^2\ dx = 1/3$$

So, the result for the clockwise direction is $\boxed{-1/3}$.

Problem 13A) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x, y, z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 1 - \cos(t) - \sin(t) \rangle$ with $t \in [0, 2\pi]$.

Solution:
The curve is contained in the graph of the function $f(x, y) = 1 - x - y$. That surface is parameterized by $\vec{r}(u, v) = \langle u, v, 1 - u - v \rangle$ and has the normal vector $\vec{r}_u \times \vec{r}_v = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \langle 1, 1, 1 \rangle$. The curl of F is $\langle 0, 0, 3x^2 + 3y^2 \rangle$ so that $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 3(x^2 + y^2)$. The surface is parameterized over the region $R = \{u^2 + v^2 \le 1\}$ and $\int \int_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2\pi} 3r^3\ d\theta dr = \boxed{\frac{3\pi}{2}}$.

Problem 14A) (10 points)

Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \mid x^2 + y^2 \le 1\}$ in the xy-plane. The surface S is oriented so that the normal vector

points upwards. Compute the flux $\int \int_S \vec{F} \cdot d\vec{S}$ of the vectorfield
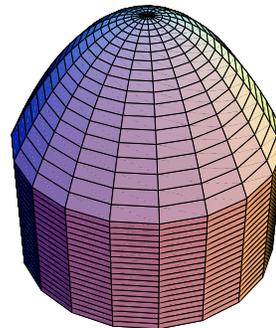
$$\vec{F} = \langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \rangle$$

through S using the divergence theorem.

Solution:

We apply the divergence theorem to the region $E = \{0 \leq z \leq f(x, y),\ x^2 + y^2 \leq 1\ \}$. Using $\text{div}(F) = -2$, we get

$$\int \int \int \text{div}(F)\ dV = \int_0^1 \int_0^{2\pi} \int_0^{2-r^2} (-2)\ r\ drd\theta dz$$

$$= (-2) \int_0^1 \int_0^{2\pi} (2 - r^2)\ r\ drd\theta dz$$

$$= (-2)(2\pi)(2/2 - 1/4) = -3\pi\ .$$

By the divergence theorem, this is the flux of F through the boundary of E which consists of the surface S, the cylinder $S_1 : r(u, v) = (\cos(u), \sin(u), v)$ with normal vector $r_u \times r_v = (-\sin(u), \cos(u), 0) \times (0, 0, 1) = (\cos(u), \sin(u), 0)$ plus the flux through the floor $S_2 : \vec{r}(u, v) = (v \sin(u), v \cos(u), 0)$ with normal vector $r_u \times r_v = (0, 0, -v)$. The flux through S_1 is

$$\int \int_{S_1} \vec{F} \cdot dS = \int_0^1 \int_0^{2\pi} F(\cos(u), \sin(u), v) \cdot (\cos(u), \sin(u), 0)\ dudv$$

$$= \int_0^1 \int_0^{2\pi} (-4\cos^2(u) + 3\sin^2(u))\ dudv = -\pi\ .$$

The flux through S_2 is

$$\int \int_{S_2} \vec{F} \cdot dS = \int_0^1 \int_0^{2\pi} F(v \sin(u), v \cos(u), 7) \cdot (0, 0, -v)\ dudv$$

$$= \int_0^1 \int_0^{2\pi} (-7v)\ dudv = -7\pi\ .$$

By the divergence theorem, $\int \int_S \vec{F} \cdot dS + \int \int_{S_1} \vec{F} \cdot dS + \int \int_{S_2} \vec{F} \cdot dS = -3\pi$ so that $\int \int_S \vec{F} \cdot dS = -3\pi + \pi + 7\pi = \boxed{5\pi}$.

PROBLEMS FOR BIO CHEM SECTIONS.

Problem 12B) (10 points)

At a county fair, two competing booths offer different games. Each charges the same price to play. At booth A, you toss a loaded coin. If you get heads for the first time on the n'th toss, you win n dollars. Let the random variable X_A be the pay-off from this game. At booth B, a trained monkey picks a point on the interval $[0, 2]$. (The monkey is honest, so the points it picks are uniformly distributed.) If it picks the point x, you win x^2. Let the random variable X_B be the payoff from this game.

Find the probability densities $p_{X_A}(x)$ and $p_{X_B}(x)$ and their expected values. Which game has a higher expected value?

Solution:
a) The expectation value is $\sum_{n=1}^{\infty} n2^{-n}$ which is $\boxed{2}$. Derivation of the summation result: use the geometric series formula $\sum_{n=0}^{\infty} a^n 1/(1-a)$ and by differentiating this formula $\sum_{n=0}^{\infty} na^{n-1} = 1/(1-a)^2$ so that $\sum_{n=0}^{\infty} na^n = a/(1-a)^2$. Whether the sum starts from $n = 0$ or $n = 1$ does not matter. To use the expectation we apply this formula for $a = 1/2$.
b) The expectation is $\int_0^2 x^2 1/2 \, dx = x^3/6|_0^2 = \boxed{8/6}$.
The first booth leads to a higher expectation to win.

Problem 13B) (10 points)

King-Kong flips a biased coin that lands heads 70% of the time. He makes 80 flips.

a) What are the expected number of heads?

b) Give an exact expression for the probability that there are 62 or more heads in this experiment. You don't have to compute the numerical value.

Solution:
a) We have a prability $p = 0.7$. The expectation of the sum of 80 independent random variables with expectation p is $80p = 56$.
b) $\sum_{k=63}^{80} \binom{k}{n} p^k (1-p)^{80-k} = 1 - \sum_{k=0}^{62} \binom{k}{n} p^k (1-p)^{80-k} = 1 - \phi_S(62)$.

Problem 14B) (10 points)

Consider an experiment which consists of throwing three dice, each with sides numbered 1-6.

a) What is the probability of getting more than 2 on at least one of the dice?

b) What is the probability that the sum showing on all three dice is less than or equal to 4.

c) Given that the first die shows 3, what is the probability that the sum of all three dice is 7?

Solution:
a) The complement event $A^c = \{(1,1,1)\}$ has probability $1/6^3$. Therefore, A has the probability $\boxed{1 - 1/6^3}$.

b) The complement event $B^c = \{(1,1,1), (1,1,2), (1,2,1), (2,1,1)\}$ has probability $4/6^3$. Therefore, B has the probability $\boxed{1 - 4/6^3}$.

c) We have to throw 3 with two dice. The event contains three cases $(1,3), (2,2), (3,1)$ and has the probability $3/36 = \boxed{1/12}$.