Name:

- Start by printing your name in the above box and check your section in the box to the left.

| MWF 10 Samik Basu |
| MWF 10 Joachim Krieger |
| MWF 11 Matt Leingang |
| MWF 11 Veronique Godin |
| TTH 10 Oliver Knill |
| TTH 115 Thomas Lam |

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 9 | | 10 |
| Total: | | 110 |

Problem 1) TF questions (20 points) No justifications needed

1) **T** | **F**

The length of the sum of two vectors is always the sum of the length of the vectors.

**Solution:**
There is a triangle inequality in general. But equality only holds for parallel vectors pointing in the same direction

2) **T** | **F**

For any three vectors, $\vec{v} \times (\vec{w} + \vec{u}) = \vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.

**Solution:**
The cross product is distributive but not commutative.

3) **T** | **F**

The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

**Solution:**
$x^2 + y^2 - z^2 = 0$ is a cone. Completion of the square adds an other constant and the surface is a one sheeted hyperboloid.

4) **T** | **F**

The functions $\sqrt{x + y - 1}$ and $\log(x + y - 1)$ have the same domain of definition.

**Solution:**
The square root is defined for 0, but the logarithm is not defined at 0.

5) **T** | **F**

If $P, Q, R$ are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing $P, Q, R$.

**Solution:**
The vectors $\vec{PQ}$ and $\vec{RQ}$ are both in the plane. The cross product is perpendicular to the plane.

6) **T** | **F**

The line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ hits the plane $2x + 3y + 4z = 9$ at a right angle.

**Solution:**
The vector $\langle 2, 3, 4 \rangle$ is in the line and perpendicular to the plane.

7) **T** F    The graph of $f(x, y) = \cos(xy)$ is a level surface of a function $g(x, y, z)$.

**Solution:**
Yes, it is the surface $g(x, y, z) = c$ for the function $g(x, y, z) = z - \cos(xy)$ and the constant $c = 0$.

8) T **F**    For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

**Solution:**
The cross product is anti commutative.

9) **T** F    If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.

**Solution:**
Assume $\vec{v}$ is not $\vec{0}$, then take $\vec{w}$ as a vector which is perpendicular to $\vec{v}$.

10) **T** F    If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

**Solution:**
The vector in question is perpendicular to $\vec{u}$ and perpendicular to $\vec{u} \times \vec{v}$. Also $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$.

11) T **F**    Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

**Solution:**
The line contains the point $(1, 1, 1)$ and a vector $\langle 2, 3, 4 \rangle$.

12) T **F** The curvature of the curve $2\vec{r}(4t)$ at $t = 0$ is twice the curvature of the curve $\vec{r}(t)$ at $t = 0$.

**Solution:**
The curvature of the first curve is $1/2$ of the curvature of the second curve.

13) T **F** The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.

**Solution:**
The surface is an elliptical cone.

14) T **F** If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.

**Solution:**
The two vectors can be parallel and nonzero.

15) T **F** Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

**Solution:**
It is parallel to $\langle 2, 3, 4 \rangle$

16) **T** F Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

**Solution:**
You can use the formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$. If this is zero, then either one of the vectors is the zero vector or $\sin(\alpha) = 0$. In all cases, this can be considered parallel.

17) **T** F The function $u(x, t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

**Solution:**
Just differentiate.

18) **T** F — Any function of three variables $f(x, y, z)$ satisfies the partial differential equation $f_{xyz} + f_{yzx} = 2f_{zxy}$.

**Solution:**
By Clairot's theorem

19) T **F** — If $f_x(x, y) = f_y(x, y)$ for all $x, y$, then $f(x, y)$ is a constant.

**Solution:**
$f_x = f_y$ is an example of a PDE called a transport equation. It has solutions like for example $f(x, y) = x + y$. Any function which stays invariant by replacing $x$ with $y$ is a solution: like $f(x, y) = \sin(xy) + x^5 y^5$.

20) T **F** — The value of the function $f(x, y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as $-0.003$.

**Solution:**
The correct approximation would be $f(0, 0) + 0.001(-1) - 0.002(2) = -0.005$.

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. No justifications are needed.

I



II



III



IV



V



VI
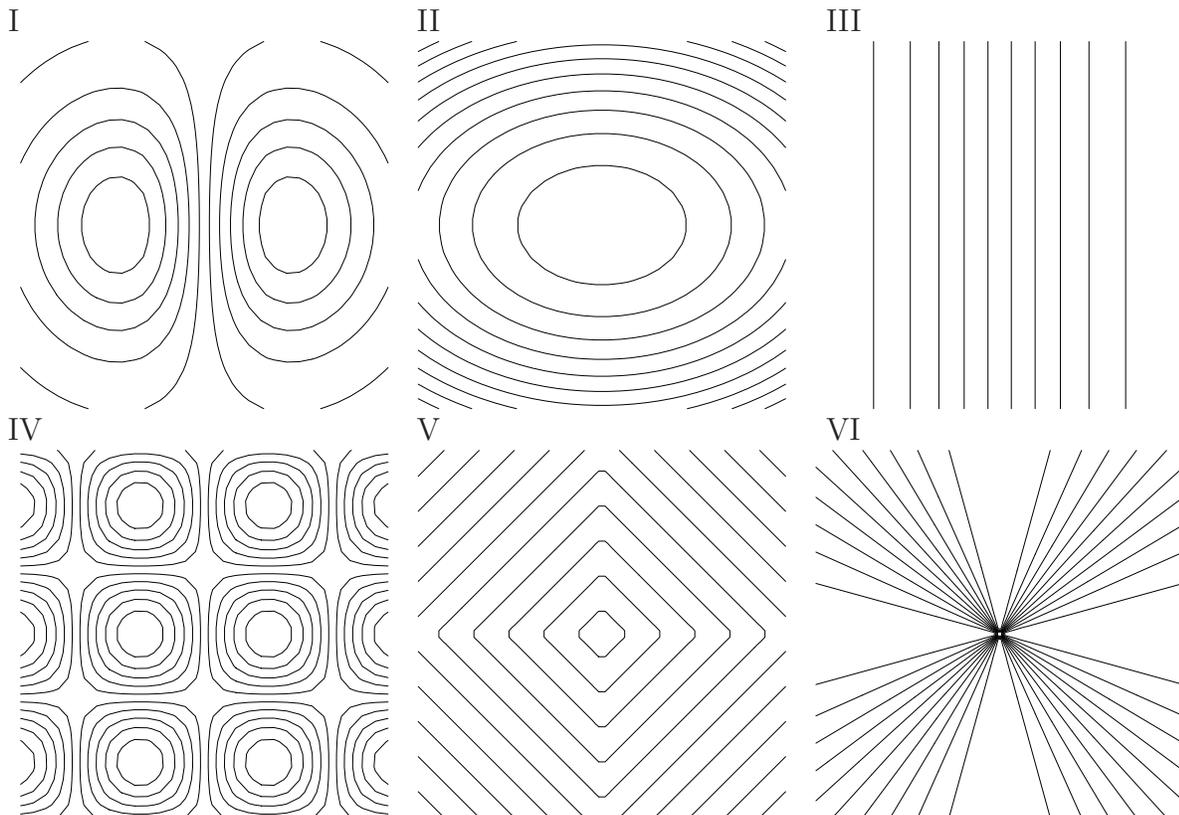


| Enter I,II,III,IV,V or VI here | Function $f(x, y)$ |
|---|---|
| | $f(x, y) = \sin(x)$ |
| | $f(x, y) = x^2 + 2y^2$ |
| | $f(x, y) = |x| + |y|$ |
| | $f(x, y) = \sin(x)\cos(y)$ |
| | $f(x, y) = xe^{-x^2 - y^2}$ |
| | $f(x, y) = x^2/(x^2 + y^2)$ |

**Solution:**

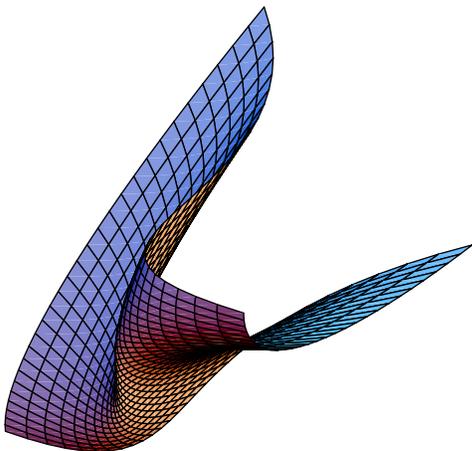| Enter I,II,III,IV,V or VI here | Function $f(x,y)$ |
| --- | --- |
| III | $f(x,y) = \sin(x)$ |
| II | $f(x,y) = x^2 + 2y^2$ |
| V | $f(x,y) = |x| + |y|$ |
| I | $f(x,y) = xe^{-x^2-y^2}$ |
| VI | $f(x,y) = x^2/(x^2+y^2)$ |

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**Problem 2b) (5 points)**

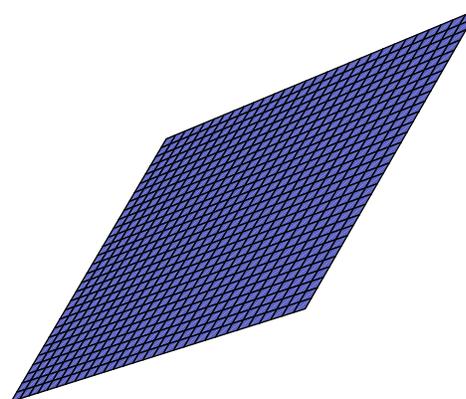Match the parametric surfaces with their parameterization. No justification is needed.
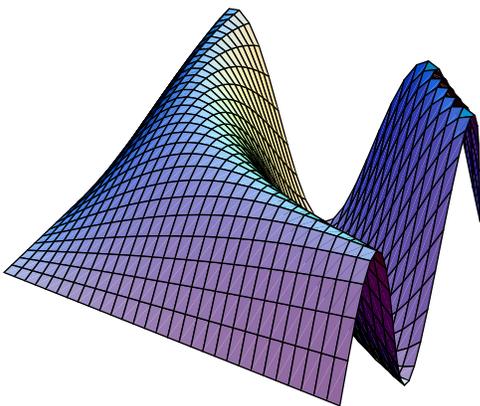
I
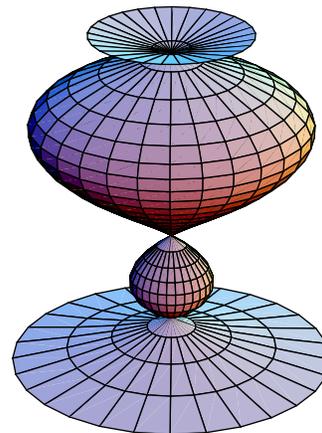
II

III

IV

| Enter I,II,III,IV here | Parameterization |
|---|---|
| | $(u, v) \mapsto (u, v, u + v)$ |
| | $(u, v) \mapsto (u, v, \sin(uv))$ |
| | $(u, v) \mapsto (0.2 + u(1 - u^2))\cos(v), (0.2 + u(1 - u^2))\sin(v), u)$ |
| | $(u, v) \mapsto (u^3, (u - v)^2, v)$ |

**Solution:**

| Enter I,II,III,IV here | Parameterization |
|---|---|
| IV | $(u, v) \mapsto (u, v, u + v)$ |
| I | $(u, v) \mapsto (u, v, \sin(uv))$ |
| II | $(u, v) \mapsto (0.2 + u(1 - u^2))\cos(v), (0.2 + u(1 - u^2))\sin(v), u)$ |
| III | $(u, v) \mapsto (u^3, (u - v)^2, v)$ |

Surface $I$ is a graph.

Surface $II$ is a surface of revolution.

Surface $III$ is algebraic. One of the traces is $(u^3, u^2)$, an other trace is the parabola $(v^2, v)$.

Surface $IV$ is a plane.

Problem 3) (10 points)

Use the technique of linear approximation to estimate $f(\log(2) + 0.001, 0.006)$ for $f(x, y) = e^{2x-y}$. (Here, log means the natural logarithm).

**Solution:**

$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$f(x_0, y_0) = e^{2\log 2} = 4$
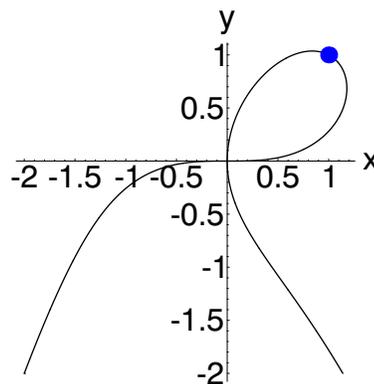
$f_x(x_0, y_0) = 8$

$f_y(x_0, y_0) = -4$

$L(x, y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.

Problem 4) (10 points)

Consider the equation

$$f(x, y) = 2y^3 + x^2y^2 - 4xy + x^4 = 0$$

It defines a curve, which you can see in the picture. Near the point $x = 1, y = 1$, the function can be written as a graph $y = y(x)$. Find the slope of that graph at the point $(1, 1)$.



**Solution:**
Use the formula for implicit differentiation which is derived from the chain rule $f_x(x, y(x))\cdot 1 + f_y(x, y(x)) \cdot y'(x) = 0$. The slope is $y'(x) = -f_x(x, y)/f_y(x, y)_{(x,y)} = (1, 1) = -1/2$.
An other possibility to solve this problem is to find the equation of the tangent line which is $f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 0$ and find the slope $m$ by writing this equation as $y = mx + b$. It gives of course the same result.

Problem 5) (10 points)

a) (6 points) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.

b) (4 points) Find the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

representing that line.

**Solution:**
a) The line of intersection has the direction $(3, -2, 1) \times (1, 2, 3) = 8(-1, -1, 1)$. The parameterization is $\boxed{\vec{r}(t) = (1, -2, 0) + t(-1, -1, 1)}$.
b) If a line contains the point $(x_0, y_0, z_0)$ and a vector $\langle a, b, c \rangle$, then the symmetric equation is

$$(x - x_0)/a = (y - y_0)/b = (z - z_0)/c .$$

In our case, where $(x_0, y_0, z_0) = (1, -2, 0)$ and $(a, b, c) = (-1, -1, 1)$, the symmetric equations are $\boxed{x - 1 = y + 2 = -z}$.

Problem 6) (10 points)

a) (4 points) Find the area of the parallelogram with vertices $P = (1,0,0)$ $Q = (0,2,0)$, $R = (0,0,3)$ and $S = (-1,2,3)$.

b) (3 points) Verify that the triple scalar product has the property $[\vec{u}+\vec{v}, \vec{v}+\vec{w}, \vec{w}+\vec{u}] = 2[\vec{u}, \vec{v}, \vec{w}]$.

c) (3 points) Verify that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ has the property
$$|[\vec{u}, \vec{v}, \vec{w}]| \leq ||\vec{u}|| \cdot ||\vec{v}|| \cdot ||\vec{w}||$$

**Solution:**
a) One has to realize which vectors form the sides of the parallelogram. The solution is $|\vec{PQ} \times \vec{PR}| = 7$.
b) $[u + v, v + w, w + u] = [u, v, w] + [u, v, u] + [u, w, w] + [u, w, u] + [v, v, w] + [v, v, u] + [v, w, w] + [v, w, u]$. Any term, where two parallel vectors appear is zero. So, only $2[u, v, w]$ remains on the right hand side.
c) Build the parallelepiped spanned by $u, v, w$ and note that one can shear it in such a way that it is contained in the box of size $||\vec{u}||$ and $||\vec{v}||$ and $||\vec{w}||$. You can also see the identity by using angle formulas for the dot product $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$ and the the length of the cross produc $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\beta)$

$$|[\vec{u}, \vec{v}, \vec{w}]| \leq |\vec{u}||\vec{v}||\vec{w}|| \cos(\alpha)|| \sin(\beta)|$$

where $\beta$ is the angle between $\vec{v}$ and $\vec{w}$ and where $\alpha$ is the angle $\vec{v} \times \vec{w}$ and $\vec{u}$.

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**Problem 7) (10 points)**

Find the distance between the two lines
$$\vec{r}_1(t) = \langle t, 2t, -t \rangle$$
and
$$\vec{r}_2(t) = \langle 1 + t, t, t \rangle .$$

**Solution:**
The point $P = (0,0,0)$ is on the first line. The point $Q = (1,0,0)$ on the second line. The vector $\vec{v} = \langle 1, 2, -1 \rangle$ in the first line and $\vec{w} = \langle 1, 1, 1 \rangle$ in the second line. We have $\vec{n} = \langle 3, -2, -1 \rangle$. Now, the distance is $3/\sqrt{14}$. $(Q - P) \cdot \vec{n}/|\vec{n}| = \langle 1, 0, 0 \rangle \cdot \langle 3, -2, -1 \rangle/|n| = 3/\sqrt{14}$.

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**Problem 8) (10 points)**

Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

**Solution:**
The line of intersection is parallel to the crossed product of $\vec{v} = \langle 2, 1, 1 \rangle$ and $\vec{w} = \langle 1, 3, 1 \rangle$ which is $\langle -2, -1, 5 \rangle$. This vector is perpendicular to the plane we are looking for. The equation of the plane is $-2x - y + 5z = 0$.

Problem 9) (10 points)

The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
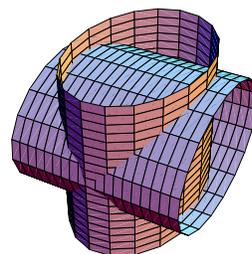c) (3 points) What is the arc length of this curve?

**Solution:**
a) Fix first $x(t), y(t)$ to satisfy the first equation then get $z(t) = \cos(t)$ by solving the second equation for $z$.
$\vec{r}(t) = (\cos(t), \sqrt{2}\sin(t), \pm\cos(t))$.
b)     We     find     the     velocity     $\vec{r}'(t)$     $=$ $(-\sin(t), \sqrt{2}\cos(t), -\sin(t))$     and     then     the     speed $|\vec{r}'(t)| = \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)} = \sqrt{2}$. The length is $\int_0^{2\pi} |\vec{r}'(t)|\ dt = \int_0^{2\pi} \sqrt{2}\ dt$. Also an expression like $\boxed{\int_0^{2\pi} \sqrt{\sin^2(t) + 2\cos^2(t) + \sin^2(t)}\ dt}$ is here correct at this stage.
c) Evaluate the integral $\boxed{2\sqrt{2}\pi}$.

Problem 10) (10 points)

a) (6 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(t), \sin(t), -2t \rangle$ at the point $\vec{r}(0)$.

b) (4 points) Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = \langle -\cos(5t), \sin(5t), -10t \rangle$ at the point $\vec{r}(0)$.

**Hint.** Use one of the two formulas for the curvature

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \,,$$

where $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$. The curvatures in b) can be derived from the curvature in a). There is no need to redo the calculation, but we need a justification.

**Solution:**
a) We use the second formula for the curvature: $\vec{r}\,'(t) = \langle \sin(t), \cos(t), -2 \rangle$. $\vec{r}\,''(t) = \langle \cos(t), -\sin(t), 0 \rangle$. The speed of the curve satisfies $|\vec{r}\,'(t)| = \sqrt{5}$. The vector $\vec{r}\,'(t) \times \vec{r}\,''(t)$ is $(-2\sin(t), -2\cos(t), -1)$ which has length $\sqrt{5}$. therefore, the curvature is constant $\boxed{\kappa(t) = 1/5}$.

b) Because the curvature is independent of the parametrization, the curvature is agin $1/5$.