Name:

MWF 10 Samik Basu
MWF 10 Joachim Krieger
MWF 11 Matt Leingang
MWF 11 Veronique Godin
TTH 10 Oliver Knill
TTH 115 Thomas Lam

- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which the grader can not read will not receive credit.
- No notes, books, calculators, computers, or other electronic aids can be used.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F The vectors $\langle 3, -2, 1 \rangle$ and $\langle -6, 4, -2 \rangle$ are parallel.

Solution:
Indeed, one is -2 times the other.

2) T **F** If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$.

Solution:
No, the vectors can be parallel without beeing zero.

3) T **F** The surface $z^2 + 4y^2 = x^2 + 1$ is a two sheeted hyperboloid.

Solution:
It is a deformed one-sheeted hyperboloid.

4) **T** F The surface $4x^2 - 4x + y^2 - 2y - 120 = -z^2$ is an ellipsoid.

Solution:
Complete the square

5) **T** F The parametrized lines $\vec{u}(t) = \langle 1 + 2t, 2 - 5t, 1 + t \rangle$ and $\vec{v}(t) = \langle 3 - 4t, -3 + 10t, 2 - 2t \rangle$ are the same line.

Solution:
The vectors are parllel, and both lines go through the same point.

6) **T** F The surface $\sin(x) = z$ contains lines which are parallel to the y-axis.

Solution:
One can translate the surface in the y direction.

7) T **F** If $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{w} = 0$ and $\vec{v}$ is not the zero vector, then $\vec{u} \cdot \vec{w} = 0$.

Solution:
The assumption means that $\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{w}$. But that does not mean that $\vec{u}$ and $\vec{w}$ are perpendicular.

8) T **F** The curvature of a curve depends upon the speed at which one travels upon it.

Solution:
The curvature does not depend on the parametrization.

9) T **F** Two lines in space that do not intersect must be parallel.

Solution:
They can be skew symmetric.

10) **T** F The intersection of the ellipsoid $x^2/3 + y^2/4 + z^2/3 = 1$ with the plane $y = 1$ is a circle.

Solution:
Just set $y = 1$ in that equation.

11) T **F** The line $\vec{r}(t) = \langle 1 + 2t, 1 + 2t, 1 - 4t \rangle$ hits the plane $x + y + z = 9$ at a right angle.

Solution:
In order to be perpendicular, the velocity vector would have to be parallel to $\langle 1, 1, 1 \rangle$.

12) T **F** A line in space can intersect an elliptic paraboloid in 4 points.

Solution:
It can only intersect it in 2 points or 1 point or avoid it at all.

13) | T | F |

There is a quadric $ax^2+by^2+cz^2+dx+ey+fz = e$ which is a hyperbola when intersected with the plane $z = 0$, which is a hyperbola when intersected with the plane $y = 0$ and which is a parabola when intersected with $x = 0$.

Solution:
The answer can be found by checking through all the quadrics $ax^2 + by^2 + cz^2 + dx + ey + fz = 0$ we have seen hyperbololids, parabololids, ellipsoids, cylinders, cones. None of them has this property. P.S. If the quadrics were allowed to be alligned differently, for example if we are allowed to turn a given quadric in space, the answer changes to yes: take a cone with openeing angle larger than 90 degrees, turn it so that the xy-plane is parallel to a tangent plane, cutting the cone in a parabola. For most translations of this cone, the other two coordinate axes cut hyperbola.

14) | T | F |

The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is always in the same plane together with $\vec{v}$ and $\vec{w}$.

Solution:
Let $\vec{n} = (\vec{v} \times \vec{w})$ be the vector perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$. Then $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \vec{n}$ is perpendicular to $\vec{n}$. It is therefore parallel to the plane.

15) | T | F |

If $\vec{u} \times \vec{v} = 0$ and $\vec{u} \cdot \vec{v} = 0$, then one of the vectors $\vec{u}$ and $\vec{v}$ is zero.

Solution:
A vector which is both parallel and perpendicular to an other vector can only be the zero vector.

16) | T | F |

If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel at time $t = 1$, then the curvature $\kappa(t)$ of the curve is zero at time $t = 1$.

Solution:
You can see this from the formula $\kappa = |r'(t) \times r''(t)|/|r'(t)|^3$. You can also think about it as follows. Assume the curvature were $\kappa = 1/r$. Then you as well locally move on a circle with radius r. But the acceleration has now a component perpendicular to your velocity vector. But we assumed there is no such acceleration.

17) T **F**

If the speed of a parametrized curve is constant over time, then the curvature of the curve $\vec{r}(t)$ is zero.

Solution:
It would be true if the velocity would be constant over time. But we can move on a circle with constant speed.

18) T **F**

The scalar projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the scalar projection of $\vec{w}$ onto $\vec{v}$.

Solution:
Look at the formula for $\text{comp}_{\vec{v}}(\vec{w})$. It is the dot product divided by the length of $\vec{v}$. If the lengths of $\vec{v}$ and $\vec{w}$ are the same, then the statement is true. In general, it is not.

19) **T** F

The value of the function $f(x, y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use formula for $L(x, y)$.

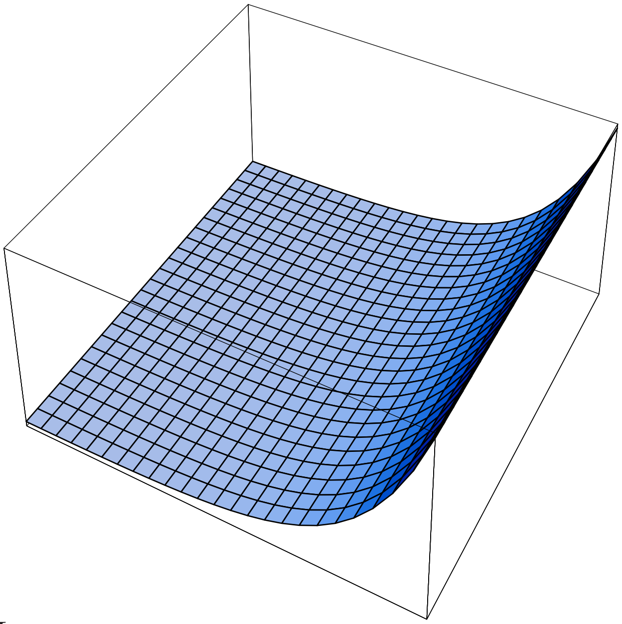
20) **T** F

The function $f(x, y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

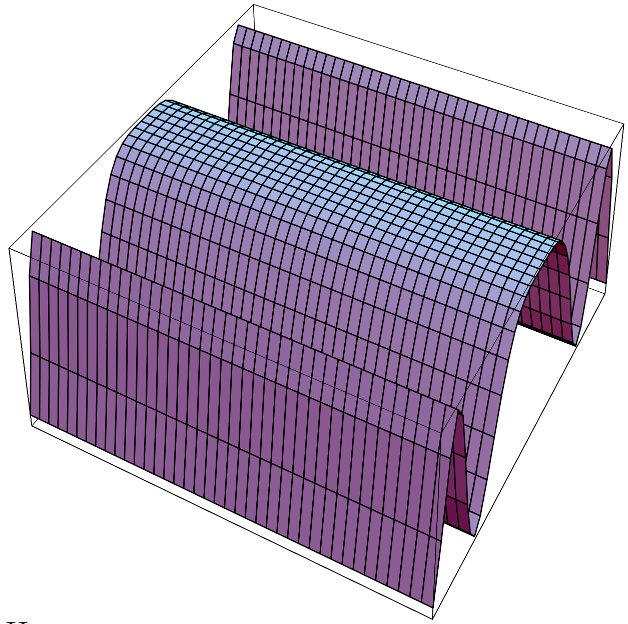
Solution:
By Clairots theorem, we can have all three x derivatives at the beginning.

Problem 2a) (2 points)

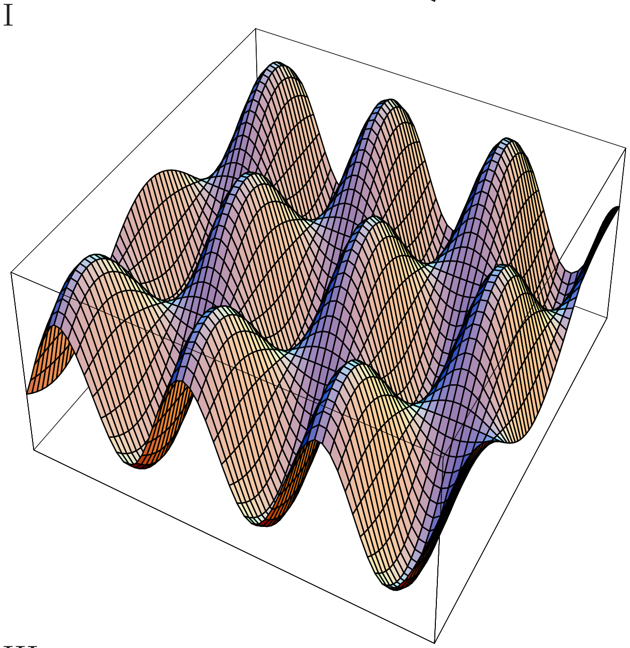
Match the equation with their graphs. No justifications are needed.



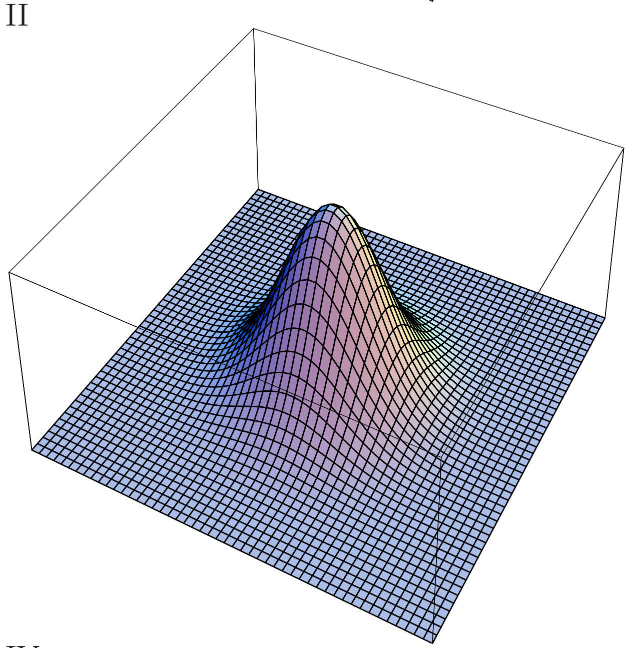
I



II



III



IV

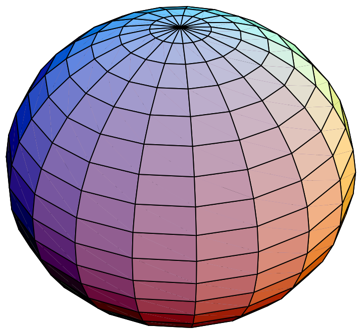
Enter I,II,III,IV here	Equation
	$z = \sin(5x)\cos(2y)$
	$z = \cos(y^2)$
	$z = e^{-x^2-y^2}$
	$z = e^x$

Solution:

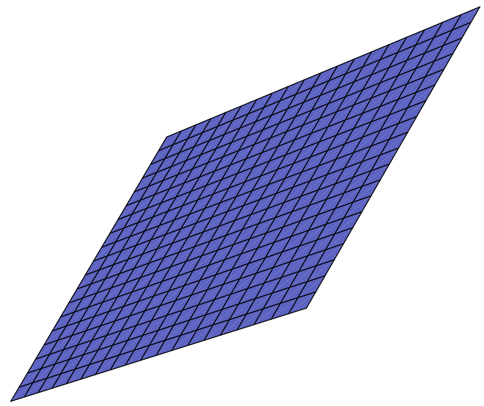
Enter I,II,III,IV here	Equation	Justification
III	$z = \sin(5x)\cos(2y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
IV	$z = e^{-x^2-y^2}$	has a maximum at (0,0)
I	$z = e^x$	no y dependence, monotone in x

Problem 2b) (3 points)

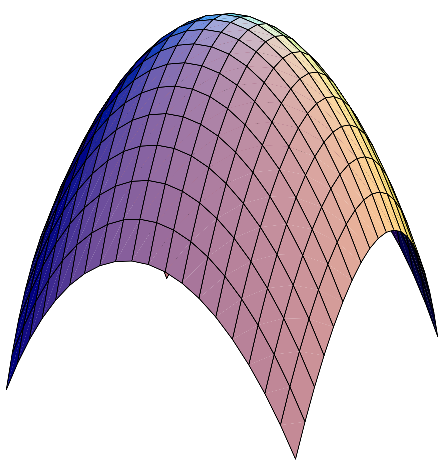
Match the parametric surfaces with their parameterization. No justification is needed.



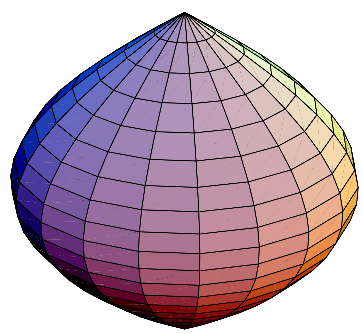
I



II



III



IV

Enter I,II,III,IV here	Parameterization
	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$
	$(u,v) \mapsto (u-1, v+3, u+v)$
	$(u,v) \mapsto (u, v, 1-u^2-v^2)$
	$(u,v) \mapsto (\sin(v)\cos(u), \sin(v)\sin(u), v)$

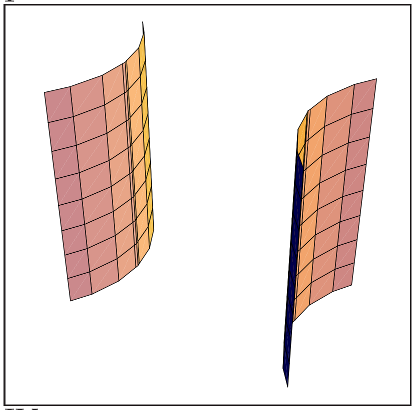
Solution:

Enter I,II,III,IV here	Parameterization
I	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$
II	$(u,v) \mapsto (u-1, v+3, u+v)$
III	$(u,v) \mapsto (u, v, 1-u^2-v^2)$
IV	$(u,v) \mapsto (\in (v)\cos(u), \sin(v)\sin(u), v)$

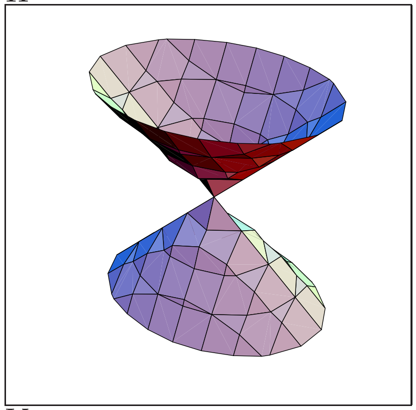
Problem 2c) (4 points)

Match the equations with the surfaces.

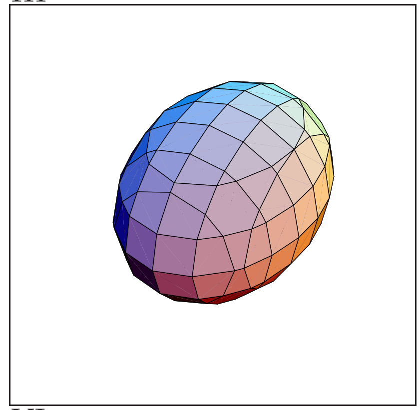
I



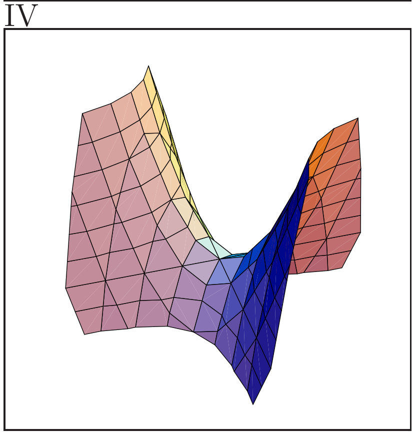
II



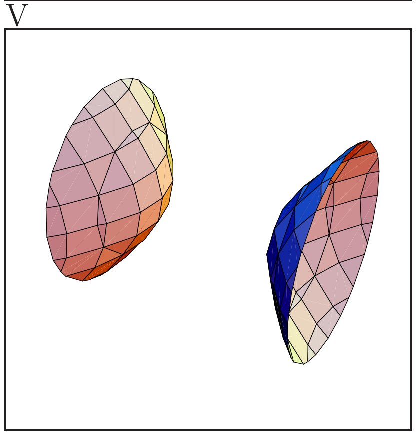
III



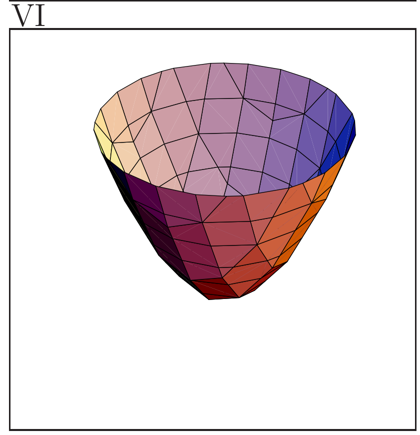
IV



V



VI



Enter I,II,III,IV,V,VI here	Equation
	$x^2 - y^2 - z^2 = 1$
	$x^2 + 2y^2 = z^2$
	$2x^2 + y^2 + 2z^2 = 1$
	$x^2 - y^2 = 5$
	$x^2 - y^2 - z = 1$
	$x^2 + y^2 - z = 1$

Solution:

Enter I,II,III,IV,V,VI here	Equation
V	$x^2 - y^2 - z^2 = 1$
II	$x^2 + 2y^2 = z^2$
III	$2x^2 + y^2 + 2z^2 = 1$
I	$x^2 - y^2 = 5$
IV	$x^2 - y^2 - z = 1$
VI	$x^2 + y^2 - z = 1$

Problem 3) (10 points)

a) Show that for any differentiable function $g(x)$, the function $u(x,y) = g(x^2 + y^2)$ satisfies the partial differential equation $yu_x = xu_y$.

b) Assuming $g'(5) \neq 0$, let u be the function defined in a). Find the unit vector $\vec{v}$ in the direction of maximal increase at the point $(x,y) = (2,1)$.

Solution:
a) Just differentiate:
$yu_x = yg'(x^2 + y^2)2x = 2xyg'(x^2 + y^2)$
$xu_y = xg'(x^2 + y^2)2y = 2yxg'(x^2 + y^2)$
These two expressions are the same.
b) The direction of maximal increase points into the direction of the gradient of u which is $\nabla u(x,y) = (g'(x^2 + y^2)2x, g'(x^2 + y^2)2y)$.
At the point $(x,y) = (2,1)$ we have $(g'(5)4, g'(5)2)$. If we normalize that, we obtain $\boxed{\vec{v} = (4,2)/\sqrt{20}}$.

Problem 4) (10 points)

a) (7 points) Find a parametric equation for the line which is the intersection of the two planes $2x - y + 3z = 9$ and $x + 2y + 3z = -7$.

b) (3 points) Find a plane perpendicular to both planes and which passes through the point $P = (1, 1, 1)$.

Solution:
a) We get the direction of the line by taking the crossed product of $\langle 2, -1, 3\rangle$ and $\langle 1, 2, 3\rangle$ which is $\langle -9, -3, 5\rangle$. To find a point in both lines, subtract one from the other to get $x - 3y = 16$. If $z = 0$, then $2x - y = 9$ and $x + 2y = -7$ so that $x = 11/5, y = -23/5$. The parametric equations are $(x, y, z) = (11/5, -23/5) + t(-9, -3, 5)$.
b) Plug in the coordinates $(x, y, z) = (1, 1, 1)$ of the point to get the constant $-9x - 3y + 5z = -7$.

Problem 5) (10 points)

Given the vectors $\vec{v} = \langle 1, 1, 0\rangle$ and $\vec{w} = \langle 0, 0, 1\rangle$ and the point $P = (2, 4, -2)$. Let Σ be the plane which goes through the origin which contains the vectors $\vec{v}$ and $\vec{w}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

a) (6 points) Compute the distance from P to the plane Σ.

b) (4 points) Find the shortest distance from P to the sphere S.

Hint for b): Find first the distance from P to the origin $O = (0, 0, 0)$.

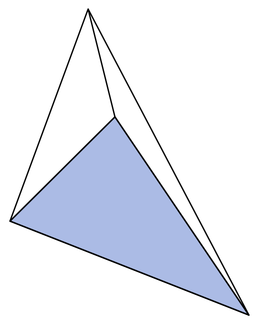
Solution:
a) $\Sigma : x - y = 0$, $n = (1, -1, 0)$. The plane is $x - y = 0$. The point $Q = (0, 0, 0)$ is on the plane. $\vec{PQ} \cdot \vec{n}/|\vec{n}| = \langle 2, 4, -2\rangle \cdot \langle 1, -1, 0\rangle/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$ is the distance.
b) $\sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$ is the distance to the origin. So the distance to the sphere is 1 less. The anser is $\sqrt{24} - 1$.

Problem 6) (10 points)

a) (6 points) Find an equation for the plane through the points $A = (0, 1, 0), B = (1, 2, 1)$ and $C = (2, 4, 5)$.

b) (4 points) Given an additional point $P = (-1, 2, 3)$, what is the volume of the tetrahedron which has A, B, C, P among its vertices.

A useful fact which you can use without justification in b): the volume of the tetrahedron is 1/6 of the volume of the parallelepiped which has AB, AC, and AP among its edges.



Solution:
a) The vectors $\vec{v} = \vec{AB} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \vec{AC} = \langle 2, 3, 5 \rangle$ are in the plane. Their cross product is $\vec{n} = \langle 2, -3, 1 \rangle$. This vector is perpendicular to the plane. The equation of the plane is therefore $2x - 3y + z = d$. Plugging in one point like A, gives $d = -3$.
b) With the vector $\vec{u} = \vec{AP} = \langle -1, 1, 3 \rangle$, one can express the volume of the parallelepiped as $|[\vec{u}, \vec{v}, \vec{w}]| = |\vec{u} \cdot \vec{n}| = |\langle -1, 1, 3 \rangle \cdot \langle 2, -3, 1 \rangle| = |2| = 2$. The volume of the tetrahedron is $2/6 = 1/3$.

Problem 7) (10 points)

The parametrized curve $\vec{u}(t) =< t, t^2, t^3 >$ (known as the "twisted cubic") intersects the parametrized line $\vec{v}(s) = \langle 1 + 3s, 1 - s, 1 + 2s \rangle$ at a point P. Find the angle of intersection.

Solution:
The curves intersect at $P = (1, 1, 1)$ with $t = 1$, $s = 0$ So, it remains to find the angle between $\vec{v} = \langle 1, 2, 3 \rangle$ and $\vec{w} = \langle 3, -1, 2 \rangle$, which is 60 degrees.

Problem 8) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (\log(t), 2t, t^2)$, where $\log(t)$ is the natural logarithm (denoted by $\ln(t)$ in some textbooks).

a) What is the velocity and what is the acceleration at time $t = 1$?

b) Find the length of the curve from $t = 1$ to $t = 2$.

Hint: you should end up with a final integral which does not involve any square roots and which you can solve.

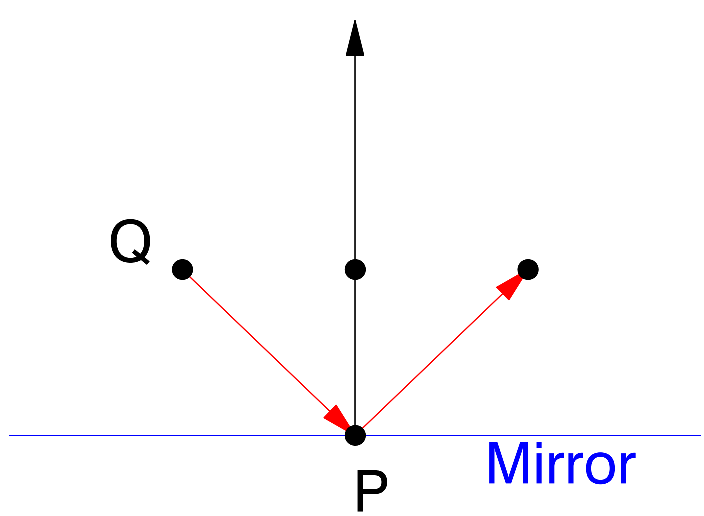
Solution:
a) $\vec{v}(t) = \vec{r}'(t) = \langle 1/t, 2, 2t \rangle$.
$\vec{v}(1) = \langle 1, 2, 2 \rangle$
$\vec{a}(t) = \vec{r}''(t) = \langle -1/t^2, 2, 6t \rangle$.
$\vec{a}(1) = \langle -1, 0, 2 \rangle$.
b) $\int_1^2 \sqrt{1/t^2 + 4 + 4t^2}\, dt = \int_1^2 1/t + 2t\, dt = \log(t) + t^2|_1^2 = \log(2) + 3$.

Problem 9) (10 points)

A planar mirror in space contains the point $P = (4, 1, 5)$ and is perpendicular to the vector $\vec{n} = \langle 1, 2, -3 \rangle$. The light ray $\vec{QP} = \vec{v} = \langle -3, 1, -2 \rangle$ with source $Q = (7, 0, 7)$ hits the mirror plane at the point P.

a) (4 points) Compute the projection $\vec{u} = \vec{\text{Proj}}_{\vec{n}}(\vec{v})$ of $\vec{v}$ onto $\vec{n}$.

b) (6 points) Identify $\vec{u}$ in the figure and use it to find a vector parallel to the reflected ray.



Solution:
a) $\text{Proj}_{\vec{n}}(\vec{v}) = \frac{(\vec{n}\cdot\vec{v})}{|\vec{n}|^2}\vec{n} = (\langle 1,2,-3\rangle \cdot \langle -3,1,-2\rangle)/14\vec{n} = (5/14)\langle 1,2,-3\rangle$.
b) With $\vec{u}$ we can get the reflected vector $\vec{w}$ because $\vec{w} - \vec{v} = -2\vec{u}$ so that $\vec{w} = \vec{v} - 2\vec{u}$. Note that $\vec{u}$ points down towards the mirror.

Problem 10) (10 points)

a) A duck swims near Watertown on the Charles river clockwise on the circle $\vec{r}(t) = \langle \cos(t), -\sin(t)\rangle$. The water temperature is given by the formula $T(x,y) = x^3e^y + y$. Find the temperature change $\frac{dT}{dt}$ the duck feels at time $t = 0$

b) The place where the temperature $T(x,y) = 1$ is constant can be writen as $y = f(x)$ near the point $(1,0)$. Find $f'(1)$.

Solution:
a) The gradient vector of T is $\nabla T(x,y) = \langle 3x^2e^y, x^3e^y + 1\rangle$. The the point $(x,y) = \vec{r}(1) = (1,0)$, we have $\nabla T(x,y) = \langle 3,2\rangle$. The velocity vector at this point is $\langle 0,-1\rangle$. The dot product is 2.
b) Implicit differentiation gives $f_x = -T_x/T_y = -3/2$.