Name:

- Start by printing your name in the above box and check your section in the box to the left.

| MWF 10 Samik Basu |
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- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | $(1,1)$ is a local maximum of the function $f(x,y) = x^2 y - x + \cos(y)$.

Solution:
$(1,1)$ is not even a critical point.

2) | T | **F** | If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

Solution:
Take $f(x,y) = x^2$ for example. Every point on the y axes $\{x = 0\}$ is a critical point.

3) | **T** | F | If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$.

4) | T | **F** | There is no function $f(x,y,z)$ of three variables, for which every point on the unit sphere is a critical point.

Solution:
Take a function like $g(t) = te^{-t}$ with a maximum at $t = 1$ and define $f(x,y,z) = g(x^2 + y^2 + z^2)$.

5) | T | **F** | If (x_0, y_0) is a maximum of $f(x,y)$ under the constraint $g(x,y) = g(x_0, y_0)$, then (x_0, y_0) is a maximum of $g(x,y)$ under the constraint $f(x,y) = f(x_0, y_0)$.

Solution:
Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

6) **T** | F

If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

7) **T** | F

The vector $\vec{r}_u - \vec{r}_v$ is tangent to the surface parameterized by $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.

Solution:
Both vectors $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface. So also their difference.

8) **T** | F

If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c, h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1, 1, 1)$ is a critical point of f.

Solution:
Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0, 0, 0)$.

9) **T** | F

If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axes the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

10) **T** | F

Let (x_0, y_0) be a saddle point of $f(x, y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

Solution:
Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steepest increase (this is the gradient direction).

11) $\boxed{\text{T}}$ $\boxed{\text{F}}$ A function $f(x, y)$ on the plane for which the absolute minimum and the absolute maximum are the same must be constant.

Solution:
This would not be true if "absolute" would be replaced by "local".

12) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The sign of the Lagrange multiplier tells whether the critical point of $f(x, y)$ constrained to $g(x, y) = 0$ is a local maximum or a local minimum.

Solution:
We would get the same Lagrange equations when replacing g with $-g$ and λ with $-\lambda$.

13) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The point $(0, 1)$ is a local minimum of the function $x^3 + (\sin(y - 1))^2$.

Solution:
While the gradient is $(3x^2, 2\sin(y - 1)\cos(y - 1))$, the critical point is not a minimum.

14) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The integral $\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} 1 \ d\theta d\phi d\rho$ is the volume of the ice cream cone obtained by intersecting $x^2 + y^2 \le z^2$ with $x^2 + y^2 + z^2 \le 1$.

Solution:
The cone is in spherical coordinates described by $0 \le \rho \le 1, 0 \le \phi \le \pi/4$. But the factor $\rho^2 \sin(\phi)$ is missing.

15) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The formula $\int_0^1 \int_0^y f(x, y) \ dxdy = \int_0^1 \int_0^x f(x, y) \ dydx$ holds for all functions $f(x, y)$.

Solution:
The two integrals describe two different regions.

16) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The surface area of a surface does not depend on the parameterization of the surface.

Solution:
This is analogous to the corresponding property for arc length.

17)
T	**F**

The formula for surface area is $\int \int_R |\vec{r}_u \cdot \vec{r}_v| \, dudv$.

Solution:
We have a cross product, not a dot product.

18)
T	**F**

The integral $\int_0^1 \int_0^1 f(x,y) \, dxdy$ is the volume under the graph of f and so non-negative.

Solution:
The integral is a signed volume. It can be negative.

19)
T	F

$\int_{-2}^2 \int_{-3}^3 (x^2 + y^2) \sin(y) \, dxdy = 0$.

Solution:
The y-integral from y on $[-2, 0]$ and the y-integral on $[0, 2]$ cancel.

20)
T	**F**

When changing to cylindrical coordinates, we include a factor $\rho^2 \sin(\phi)$.
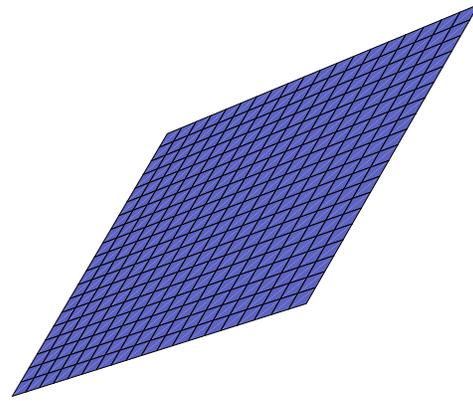
Solution:
This would be the correct factor if we were to change to spherical coordinates.
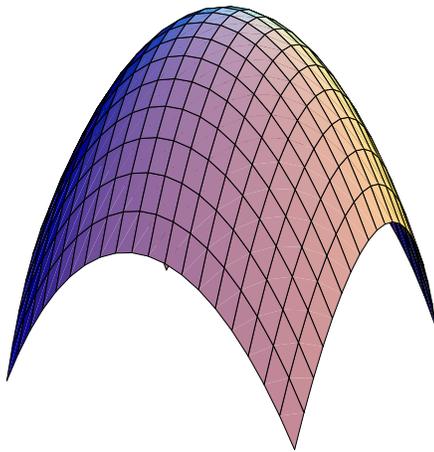

Problem 2) (10 points)

Match the parametric surfaces $S = \vec{r}(R)$ with the corresponding surface integral $\int \int_S \, dS = \int \int_R |\vec{r}_u \times \vec{r}_v| \, dudv$. No justifications are needed.
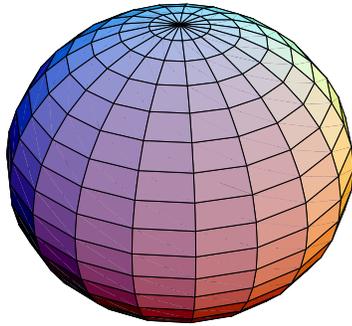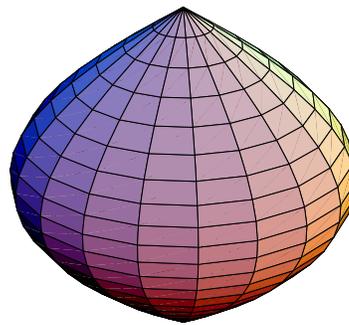
I

II

III

IV

Enter I,II,III,IV here	Surface integral
	$\int\int_R |\vec{r}_u \times \vec{r}_v|\, dudv = \int_0^1 \int_0^1 \sqrt{1+4u^2+4v^2}\, dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v|\, dudv = \int_0^1 \int_0^1 \sqrt{3}\, dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v|\, dudv = \int_0^{2\pi} \int_0^\pi \sin(v)\sqrt{1+\cos(v)^2}\, dvdu$
	$\int\int_R |\vec{r}_u \times \vec{r}_v|\, dudv = \int_0^{2\pi} \int_0^\pi \sin(v)\, dvdu$

Solution:
Surface I is a graph for which we had derived a formula. Surface II is a surface of revolution. Also this formula had been derived. Surface III is algebraic. One of the traces is (u^3, u^2), an other trace is the parabola (v^2, v). Surface IV is a plane.

Enter I,II,III,IV here	Surface integral
III	$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv = \int_0^1 \int_0^1 \sqrt{1 + 4u^2 + 4v^2}\ dvdu$
II	$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv = \int_0^1 \int_0^1 \sqrt{3}\ dvdu$
IV	$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv = \int_0^{2\pi} \int_0^\pi \sin(v)\sqrt{1 + \cos(v)^2}\ dvdu$
I	$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv = \int_0^{2\pi} \int_0^\pi \sin(v)\ dvdu$

Problem 3) (10 points)

Find and classify all the critical points of the function $f(x, y) = xy(4 - x^2 - y^2)$.

Solution:
The gradient of $f(x, y) = 4xy - x^3y - xy^3$ is $\nabla f(x, y) = \langle 4y - 3x^2y - y^3, 4x - x^3 - 3xy^2 \rangle$.
We solve the system

$$y(4 - 3x^2 - y^2) = 0 \tag{1}$$
$$x(4 - x^2 - 3y^2) = 0 \tag{2}$$

There are four possibilities:
1) $y = 0, x = 0$
2) $4 - 3x^2 - y^2 = 0, x = 0$
3) $4 - x^2 - 3y^2 = 0, y = 0$
4) $4 - 3x^2 - y^2 = 0, 4 - x^2 - 3y^2 = 0$.
This gives 7 critical points in total
1) gives the critical point $(0, 0)$.
2) gives the critical points $(0, 2), (0, -2)$.
3) gives the critical points $(2, 0), (-2, 0)$.
4) (subtract 3 times the second equation from the first): $(1, 1), (-1, 1), (1, -1), (-1, -1)$.
The discriminant $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ at a general point is $-9(x^4 + y^4) - 16 + 24(x^2 + y^2) + 18x^2y^2$ and $f_{xx}(x, y) = -6xy$.
Applying the second derivative test gives

Critical point	(-2, 0)	(-1, -1)	(-1, 1)	(0, 0)	(1, -1)	(1, 1)	(2, 0)	(-2,0)	(2,0)
Discriminant	-64	32	32	-16	32	32	-64	-64	-64
f_{xx}	0	-6	6	0	6	-6	0	0	0
Analysis	saddle	max	min	saddle	min	max	saddle	sadle	saddle

Problem 4) (10 points)

Find the area of the moon-shaped region outside the disc of radius 1 and inside the Cardiod $r = 1 + \cos(\theta)$.

Solution:
The interval for which $1 + \cos(\theta)$ is positive is $[-\pi/2, \pi/2]$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos(\theta)} r \, dr d\theta \ .$$

The inner integral is $\int_1^{1+\cos(\theta)} r \, dr = (1 + \cos(\theta))^2/2 - 1/2$. Because

$$\int_{-\pi/2}^{\pi/2} 1/2 + \cos(\theta) + \cos(\theta)^2/2 \, d\theta = \pi/2 + \pi/4 + 2 \ .$$

The result is $\boxed{3/2 + 3\pi/4}$.

Problem 5) (10 points)

Minimize the function $E(x, y, z) = \frac{k^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$ under the constraint $xyz = 8$, where k^2 and m are constants.

Remark. In quantum mechanics, E is the ground state energy of a particle in a box with dimensions x, y, z. The constant k is usually denoted by $\hbar$ and called the Planck constant.

Solution:
Write $C = k^2/(8m)$ to save typing. $\nabla E(x, y, z) = -2C(1/x^3, 1/y^3, 1/z^3)$. The constraint is $G(x, y, z) = xyz - 8 = 0$. We have $\nabla G(x, y, z) = (yz, xz, xy)$. The Lagrange equations are
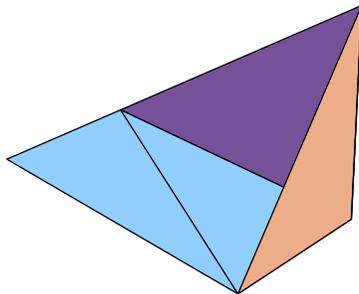
$$\begin{aligned}
2C &= \lambda x^3 yz \\
2C &= \lambda xy^3 z \\
2C &= \lambda xyz^3 \\
xyz &= 8
\end{aligned}$$

Eliminating λ gives $x^2 = y^2 = z^2$ and $x = y = z = 2$ and the minimal energy is $\boxed{3C/4 = 3k^2/(32m)}$.

Problem 6) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular

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to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?

Solution:

The Lagrange equations are

$$
\begin{aligned}
bc &= \lambda(b + c/2) \\
ac &= \lambda(a + c/2) \\
ab &= \lambda(a + b)/2 \\
ab + bc/2 + ac/2 &= 12 .
\end{aligned}
$$

Dividing the first to the second equation leads to $a = b$. Dividing the second to the third equation gives $c = 2b$. Substituting a and c gives $b^2 + b^2 + b^2 = 12$ or $b = 2$. Therefore $a = 2, b = 2, c = 4$ is the optimal configuration. The maximal volume is $f(2, 2, 4) = 8/3$.

Problem 7) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta$ for $\theta \in [0, \pi/2]$. Find the double integral

$$
\iint_R \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}(\pi/2 - \sqrt{x^2 + y^2})} \, dx \, dy .
$$
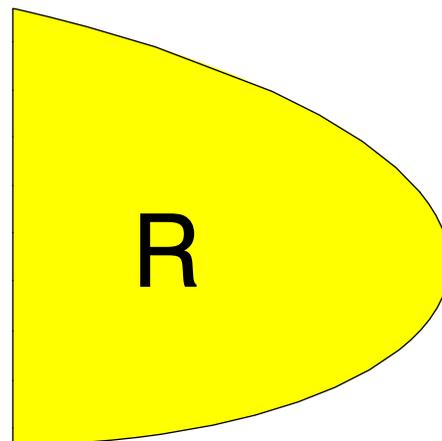
Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = dr d\theta$ does not work. The integral $\int_0^{\pi/2} \int_0^\theta \frac{\sin(r)}{r(\pi/2-r)} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^{\pi/2} \int_r^{\pi/2} \frac{\sin(r)}{r(\pi/2 - r)} r \ d\theta \ dr$$

Evaluating the inner integral gives $\int_0^{\pi/2} \sin(r) \ dr = \boxed{1}$.

Problem 8) (10 points)

Find the volume of the solid bound by $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$ above the cone $x^2 + y^2 = z^2$ and in the first octant $x \geq 0, y \geq 0, z \geq 0$.

Solution:
In spherical coordinates, the solid is the rectangular region $2 \leq \rho \leq 3, \phi \in [0, \pi/4]$ and $\theta \in 0, \pi/2$. The integral is

$$\int_2^3 \int_0^{\pi/2} \int_0^{\pi/4} \rho^2 \sin(\phi) \ d\phi d\theta d\rho = \frac{(27 - 8)}{3}(1 - \sqrt{2}/2)(\pi/2) \ .$$

Problem 9) (10 points)

Find $\int \int \int_R z^2 \ dV$, where R of the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.

Solution:
Since the result for the double cone is just twice the result for the single cone, we work with the region R obtained with the single cone and multiply at the end with 2. In spherical coordinates, the solid R is as $1 \le \rho \le 2$ and $0 \le \phi \le \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi) \; d\phi d\theta d\rho = (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos(\phi))}{3}|_0^{\pi/4} = 2\pi(31/5)(1 - 2^{-3/2}) \,.$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 2^{-3/2})}$.

Problem 10) (10 points)

Consider the region inside $x^2 + y^2 + z^2 = 2$ above the surface $z = x^2 + y^2$.

a) Sketch the region.

b) Find its volume.

Solution:

a) The intersection of the two surfaces is a circle of radius 1. The region is the bottom of a paraboloid covered with a spherical cap.
b) Use cylindrical coordinates: $2\pi \int_0^1 (\sqrt{2 - r^2} - r^2) r \, dr =$
$-(\pi/3)(2 - r^2)^{3/2}|_0^1 - \pi/2 = \boxed{(\pi/3)(2^{3/2} - 1) - \pi/4}$.