Name:

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- Start by printing your name in the above box and check your section in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (30 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** $f(x,y)$ and $g(x,y) = f(x^2, y^2)$ have the same critical points.

Solution:
The function g has always $(0,0)$ as a critical point, even if f has not.

2) **T** F If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

Solution:
At a critical point the gradient is $(a,b) = (0,0)$, which implies $f = 0$.

3) **T** F Given 2 arbitrary points in the plane, there is a function $f(x,y)$ which has these points as critical points and no other critical points.

Solution:
Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates (-1,0),(1,0). The function $f(x,y) = -y^2(x^3 - 1)$ has the two points as critical points. One is a local max, the other is a saddle point.

4) T **F** If (x_0, y_0) is the maximum of $f(x,y)$ on the disc $x^2+y^2 \leq 1$ then $x_0^2+y_0^2 < 1$.

Solution:
The maximum could be on the boundary.

5) T **F** There are no functions $f(x,y)$ for which every point on the unit circle is a critical point.

Solution:
There are many rotationally symmetric functions with this property.

6) T F

An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

Solution:
The Lagrange multiplier vanishes in this case.

7) T F

If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution:
Look at a camel type surface. It has a saddle between the local maxima.

8) T F

There exists a function $f(x, y)$ of two variables which has no critical points at all.

Solution:
True. Every non-constant linear function for example.

9) T F

If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:
False, f could be constant.

10) T F

$(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

Solution:
$(0, 0)$ is a saddle point.

11) T F

If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

Solution:
Adding $x^4 + y^3$ does not change the first and second derivatives.

12) | T | **F** | Every critical point (x, y) of a function $f(x, y)$ for which the discriminant D is not zero is either a local maximum or a local minimum.

Solution:
The second derivative test give for negative D that we have a saddle point.

13) | **T** | F | If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) < 0$ then $(0, 0)$ can not be a local minimum.

Solution:
If $f_{xx}(0, 0) < 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local maximum. This means that there are points close to $(0, 0)$ where the value of f is larger.

14) | **T** | F | In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

15) | **T** | F | The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

Solution:
The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

16) | **T** | F | It is possible to find a function of two variables which has no maximum and no minimum.

Solution:
There are many linear functions like that.

17) | **T** | F | $\int_0^2 \int_0^2 (x^2 + y^2) \cos(x^3 + y^3) \, dxdy \leq 32$.

Solution:
Indeed, $x^2 + y^2 \leq 8$ on that rectangle, which has area 4.

18) T F $\int_0^2 \int_0^{x^2} f(x,y)\, dydx = \int_0^4 \int_{\sqrt{y}}^2 f(x,y)\, dxdy$.

Solution:
This is the correct change of the order of integration as you can see by drawing a picture of the region.

19) T F The area of a polar region $0 \leq r \leq r(\theta)$ is $\int_0^{2\pi} r(\theta)^2/2\, d\theta$.

Solution:
Indeed, the area is $\int_0^{2\pi} \int_0^{r(\theta)} r\, drd\theta$.

20) T F If R is the unit disc in the xy-plane, then $\int\int_R -\sqrt{1-x^2-y^2}\ dxdy = -2\pi/3$.

Solution:
It is the negative of the volume of the hemisphere of radius 1.

Problem 2) (10 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x,y)\, dxdy$
	$\int_0^1 \int_y^1 f(x,y)\, dxdy$
	$\int_0^1 \int_0^{1-y} f(x,y)\, dxdy$
	$\int_0^1 \int_0^y f(x,y)\, dxdy$

Solution:

Enter I,II,III,IV or V here.	Integral
V	$\int_0^1 \int_{1-y}^1 f(x,y)\, dxdy$
I	$\int_0^1 \int_y^1 f(x,y)\, dxdy$
II	$\int_0^1 \int_0^{1-y} f(x,y)\, dxdy$
III	$\int_0^1 \int_0^y f(x,y)\, dxdy$

I) $\int_0^1 \int_0^x f(x,y)\, dydx$

II) $\int_0^1 \int_0^{1-x} f(x,y)\, dydx$

III) $\int_0^1 \int_x^1 f(x,y)\, dydx$

IV) $\int_0^1 \int_0^{x-1} f(x,y)\, dydx$

V) $\int_0^1 \int_{1-x}^1 f(x,y)\, dydx$

Problem 3) (10 points)

Which point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ is closest to the origin?

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
There is an interesting twist to this problem (as noted by one of the students Jacob Aptekar): consider the points $(x,y,z) = (1,-1/n,8/n)$, where n is a large integer, One can check that these points lye on the surface $g(x,y,z) = 1$. Their distance to the origin however decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 4) (10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum among them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2-3, 3y^2-12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

Problem 5) (10 points)

Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x,y) = (x^4 - x, (y^2 - 1)) = (0,0)$ so that the critical points are $(0,1), (0,-1), (1,1), (1,-1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = 4x^3 - 1$.

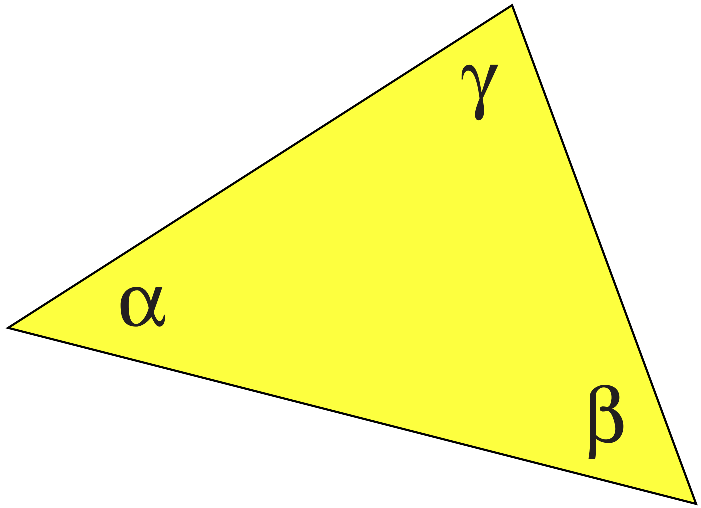
Point	D	f_{xx}	type
$(0,1)$	$D=-2$	-	saddle
$(0,-1)$	$D=2$	-1	local max
$(1,1)$	$D=6$	3	local min
$(1,-1)$	$D=-6$	-	saddle

Problem 6) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha,\beta,\gamma) = \log(\sin(\alpha)\sin(\beta)\sin(\gamma))$$

is maximal?



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.

Problem 7) (10 points)

Evaluate the integral

$$\int_{\pi/4}^{3\pi/4} \int_{1/\sin(\theta)}^{2\sin(\theta)} r \; dr d\theta$$

Hint: There is not much to compute if you know how the region looks like.

Solution:
In polar coordinates, this is a type I region. The inner curve $r(\theta) = 1/\sin(\theta)$ is equivalent to $y = r\sin(\theta) = 1$, the outer curve is $r(\theta) = 2\sin(\theta)$ which is equivalent to $x^2 + y^2 = r^2 = 2y$ or $x^2 + y^2 - 2y + 1 = 1$ which shows that it is an arc of a circle of radius 1 centered at $(0, 1)$. The region is a half disk of radius 1 with area $\pi/2$.

Problem 8) (10 points)

Evaluate the integral

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dydx \; .$$

Solution:
The region R is a triangle. A change of the order of integration gives

$$\int_0^\pi \int_0^y \frac{\sin(y)}{y}\, dxdy = \int_0^\pi \sin(y)\, dy = 0$$

Problem 9) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u,v) = \langle u, v, 2u - v \rangle$$

with $1 \leq u \leq 2$, $-1 \leq v \leq 1$.

Solution:
Compute $\vec{r}_u = \langle 1, 0, 2 \rangle$, $\vec{r}_v = \langle 0, 1, -1 \rangle$ and $\vec{r}_u \times \vec{r}_v = \langle -2, 1, 1 \rangle$. We have $|\vec{r}_u \times \vec{r}_v| = \sqrt{6}$.
The surface area is

$$\int_{-1}^1 \int_1^2 \sqrt{6}\, dudv = 2\sqrt{6} \; .$$

Problem 10) (10 points)

Integrate the function $f(x,y,z) = x^2 + y^2$ over the solid bound by the planes $z = 1, z = -1$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution:
We use cylindrical coordinates:

$$\int_{-1}^1 \int_0^{\sqrt{z^2+1}} \int_0^{2\pi} r^3 \; d\theta drdz$$