Name:

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- Start by printing your name in the above box and check your section in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Justify all your answers for problems 4-9. As usual, the path to the answer is always graded too.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- All unspecified functions are differentiable as many times as necessary.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F $\int\int\int_R 1 \; dxdydz$ is the volume of R.

Solution:
This is a basic fact.

2) T **F** At a critical point of a function f, the gradient vector has length 1.

Solution:
The gradient vector is the zero vector there.

3) T **F** At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

Solution:
The tangent plane is horizontal there.

4) **T** F For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x, y)$ is nonzero.

Solution:
Take a vector vector is perpendicular to the gradient, the directional derivative $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ is zero.

5) **T** F If $f_{xx}(0, 0) = 0, D = f_{xx}f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0, 0) = \langle 0, 0 \rangle$, then $(0, 0)$ is a saddle point.

Solution:
Because $f_{xx} = 0$, we have $D = f_{xx}f_{yy} - f_{xy}^2 = -f_{xy}^2$ which can not be positive. Because $D \neq 0$, we must have $D < 0$. By the second derivative test, the critical point is a saddle point.

6) **T** F A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

Solution:
The maximum can be either in the interior or at the boundary.

7) [T] [F] — The function $f(x,y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0,0)$.

Solution:
It is a saddle point.

8) [T] [F] — If (x,y) is a maximum of $f(x,y)$ under the constraint $g(x,y) = 5$ then it is also a maximum of $f(x,y) + g(x,y)$ under the constraint $g(x,y) = 5$.

Solution:
Indeed, on the constraint curve, the function $f + g$ is just $f + 5$, which has the same maxima and minima as f on that curve.

9) [T] [F] — The functions $f(x,y)$ and $g(x,y) = (f(x,y))^6$ always have the same critical points.

Solution:
The gradient of g is $6f^5(x,y)\nabla f$. So, the second function has critical points, where f vanishes.

10) [T] [F] — For $f(x,y,z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1,1,1)$ is perpendicular to the surface $f(x,y,z) = 4$ at the point $(1,1,1)$.

Solution:
This is a basic property of gradients.

11) [T] [F] — $f(x,y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

Solution:
The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

12) T **F** If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

Solution:
The discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ can be positive. An example is $f(x, y) = 100x^2 + 100y^2 - xy$.

13) T **F** The vector $\vec{r}_v(u, v)$ of a parameterized surface $(u, v) \mapsto \vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is always perpendicular to the surface.

Solution:
The vector is always tangent to the surface.

14) **T** F Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

Solution:
The Lagrange equations tell that $\nabla f(x, y) = (0, 0)$.

15) **T** F At a saddle point, all directional derivatives are zero.

Solution:
Because $\nabla f(x, y) = (0, 0)$ at a saddle point, all directional derivatives $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ are zero.

16) T **F** The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

Solution:
This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

17) **T** F At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

Solution:
Indeed, at a local maximum, $f_{yy} \leq 0$.

18) | T | **F** | The volume of a sphere of radius 1 is equal to the volume under the graph of $f(x,y) = \sqrt{1-x^2-y^2}$ inside the unit disc $x^2+y^2 \leq 1$.

Solution:
The integral is the volume of the half sphere.

19) | **T** | F | $\int_0^1 \int_0^1 (x^2+y^2)\, dxdy = 2/3$.

Solution:
Just integrate

20) | T | **F** | The function $f(x,y) = \int_0^x \int_0^y g(u) + g(v)\, dudv$ has the critical points (t,t), where t is a root of g.

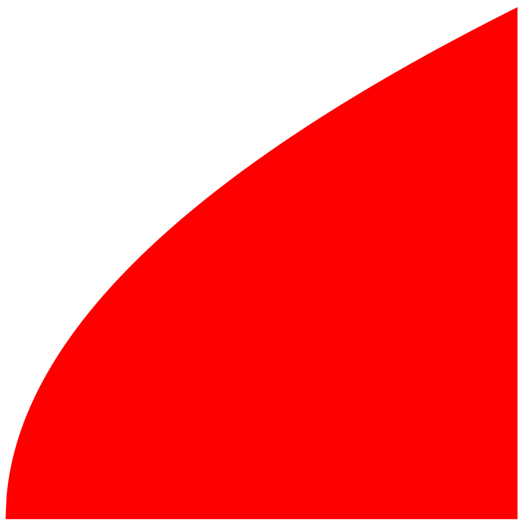
Solution:
By the fundamental theorem of calculus the gradient of f is $\nabla f(x,y) = (yg(x), xg(y))$.

Problem 2) (10 points)

Match the regions R with the corresponding double integral $\int\int_R f(x,y)\, dxdy$. No justifications are needed for this problem.

I

II



III

IV

Enter I,II,III,IV here	Integral
	$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)\, dydx$
	$\int_0^1 \int_{x^2}^1 f(x,y)\, dydx$
	$\int_0^1 \int_0^{\sqrt{x}} f(x,y)\, dydx$
	$\int_0^1 \int_0^{x^2} f(x,y)\, dydx$

Enter I,II,III,IV here	Integral
	$\int_0^1 \int_{y^2}^1 f(x,y)\, dxdy$
	$\int_0^1 \int_0^{y^2} f(x,y)\, dxdy$
	$\int_0^1 \int_{\sqrt{y}}^1 f(x,y)\, dxdy$
	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\, dxdy$

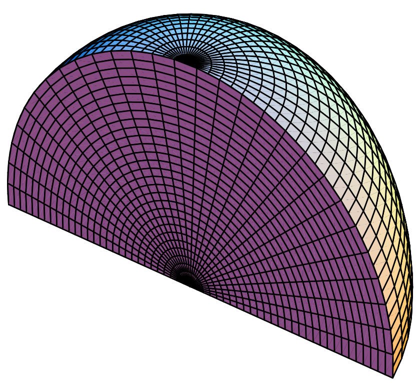
Solution:

Enter I,II,III,IV here	Integral
IV	$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)\ dydx$
II	$\int_0^1 \int_{x^2}^1 f(x,y)\ dydx$
III	$\int_0^1 \int_0^{\sqrt{x}} f(x,y)\ dydx$
I	$\int_0^1 \int_0^{x^2} f(x,y)\ dydx$

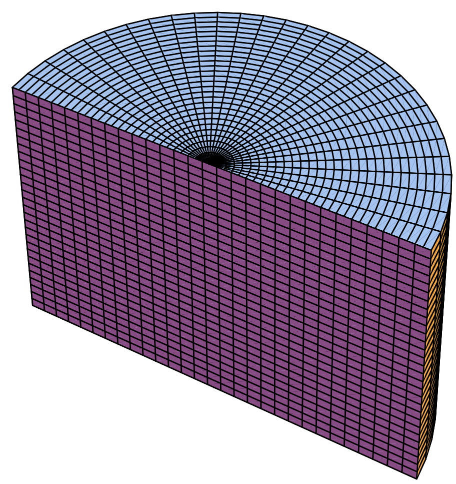
Enter I,II,III,IV here	Integral
III	$\int_0^1 \int_{y^2}^1 f(x,y)\ dxdy$
IV	$\int_0^1 \int_0^{y^2} f(x,y)\ dxdy$
I	$\int_0^1 \int_{\sqrt{y}}^1 f(x,y)\ dxdy$
II	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\ dxdy$

Problem 3) (10 points)

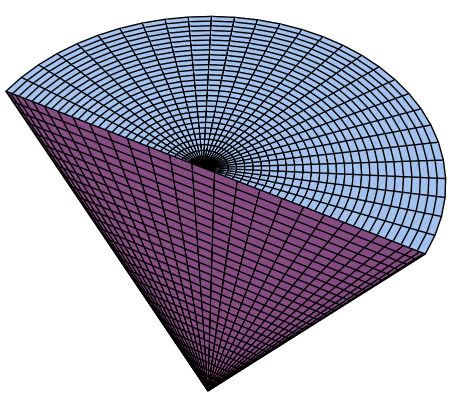
Match the solids E with the corresponding triple integral $\int\int\int_E f\ dV$. There is one triple integral, for which no picture of the solid is given. Mark this triple integral with O. No justifications are needed for this problem.



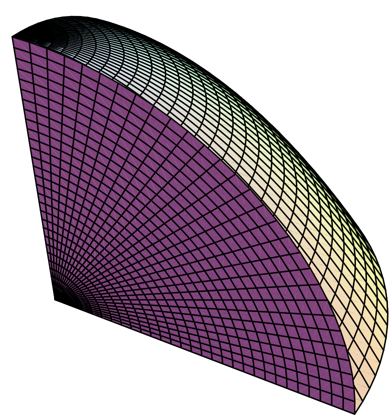
I



II



III



IV

Enter O, I,II,III or IV here	Integral
	$\int_0^1 \int_0^{\pi/2} \int_0^{1-x^2-y^2} f(x,y,z)\ rdzd\theta dr$
	$\int_0^1 \int_0^{\pi} \int_0^{\pi/2} \rho^2 \sin(\phi) f(\rho,\theta,\phi)\ d\phi d\theta d\rho$
	$\int_0^1 \int_0^{z} \int_0^{\pi/2} f(r,\theta,z)\ r\ d\theta drdz$
	$\int_0^1 \int_0^{z} \int_0^{\pi} f(r,\theta,z)\ r\ d\theta drdz$
	$\int_0^1 \int_0^{\pi} \int_0^{1} f(r,\theta,z)\ r\ drd\theta dz$

Solution:

Enter O,I,II,III or IV here	Integral
IV	$\int_0^1 \int_0^{\pi/2} \int_0^{1-x^2-y^2} f(x,y,z)\ rdzd\theta dr$
I	$\int_0^1 \int_0^{\pi} \int_0^{\pi/2} \rho^2 \sin(\phi) f(\rho,\theta,\phi)\ d\phi d\theta d\rho$
O	$\int_0^1 \int_0^{z} \int_0^{\pi} f(r,\theta,z)\ r\ d\theta drdz$
III	$\int_0^1 \int_0^{z} \int_0^{\pi} f(r,\theta,z)\ r\ d\theta drdz$
II	$\int_0^1 \int_0^{\pi} \int_0^{1} f(r,\theta,z)\ r\ drd\theta dz$

Problem 4) (10 points)

Find all the critical points of the function $f(x,y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$.
For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Solution:
$\nabla f(x,y) = \langle y^3 - x, 3xy^2 - 3y \rangle$. This is zero if $3y - 3y^5 = 0$ or $y(1-y^4) = 0$ which means $y = 0$ or $y = \pm 1$. In the case $y = 0$, we have $x = 0$. In the case $y = 1$, we have $x = 1$, in the case $y = -1$, we have $x = -1$. The critical points are $(0,0), (1,1), (-1,-1)$.
The discriminant is $f_{xx}f_{yy} - f_{xy}^2 = 3 - 6xy9y^4$. The entry f_{xx} is -1 everywhere.
Applying the second derivative test gives

Critical point	(0,0)	(1,1)	(-1,-1)
Discriminant	3	-12	-12
f_{xx}	-1	-1	-1
Analysis	max	saddle	saddle

Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x,y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Solution:
a) Lets find the critical points and classify them. Setting the gradient to 0 gives

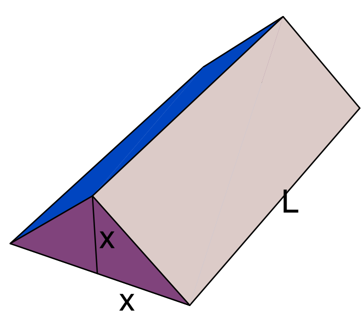
$$\begin{aligned} f_x &= 3e^y - 3x^2 = 0 \\ f_y &= 3xe^y - 3e^{3y} = 0 \ . \end{aligned}$$

The first equation gives $x = \pm e^{y/2}$. Pluging it into the second gives $y = 0, x = 1$ Applying the second derivative test $f_{xx} = -6$ and $D > 0$ shows that $(1,0)$ is a local maximum.
b) If we look at the function f restricted to the x axes, we have $g(x) = f(x,0) = 3x - 1 - x^3$. This goes to $+\infty$ for $x \to -\infty$ and goes to $-\infty$ for $x \to \infty$. We have no global maximum nor a global minimum for f.
Remark: this is a remarkable example. In single variable calculus, sometimes the statement is proven, that if one has a local maximum and no global maximum nor minimum for a function $f(x)$, then there also must exist at least one local minimum. The example here shows that this is not the case for funcations for several variables.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x,y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.

Note: this problem can also be solved by substituting one of the variables in the constraints. If done properly, such a solution needs more work. **No credit** is given for such a substitution solution in this problem.



Solution:
The Lagrange equations

$$\nabla f = \lambda \nabla g, g = 1$$

are

$$\begin{aligned} 4x + 4y &= \lambda 2xy \\ 4y &= \lambda x^2 \\ xy^2 &= 1 \end{aligned}$$

Eleminating λ gives $(4x + 4y)/4x = \lambda 2xy/\lambda x^2$ or $1 + y/x = 2y/x$ so that $1 = y/x$. The only critical point with positive x, y is $(1, 1)$. The minimum of f is $f(1, 1) = 6$. The minimal surface area is $6\sqrt{2}$.

Problem 7) (10 points)

Evaluate the double integral

$$\int_0^{27} \int_{x^{1/3}}^{3} \frac{1}{1 + y^4} \, dydx \; .$$

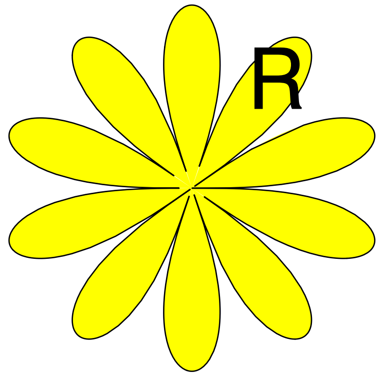
Solution:
Make a picture and change the order of integration:

$$\int_0^3 \int_0^{y^3} \frac{dxdy}{y^4 + 1}$$

This is $\log(y^4 + 1)/4|_0^3 = \log(81 + 1)/4 = \boxed{\log(82)/4}$.

Problem 8) (10 points)

The flower type region R is given in polar coordinates as $0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{|\sin(5\theta)|}$. Compute the moment of inertia $I = \int\int_R r^2 \, dA$ of this region.



Solution:
$\int_0^{2\pi} r(\theta)^4/4 \; d\theta = \int_0^{2\pi} \sin^2(\theta) \; d\theta/4 = \pi/4$. (To compute $\int_0^{2\pi} \sin(5t)^2 \; dt = \pi$, use the identity $\sin(5t)^2 = (1-\cos(10t))/2$ and note that $\int_0^{2\pi} \cos(10t) dt = 0$).

Problem 9) (10 points)

A solid E in space is given by the inequalities $x^2 + y^2 + z^2 \leq 1$, $x^2 + y^2 \geq z^2$. (In other words, the solid is obtained by cutting away the double cone $x^2 + y^2 \leq z^2$ from the unit ball $x^2 + y^2 + z^2 \leq 1$.) Compute the triple integral

$$\int \int \int_E (x^2 + y^2 + z^2) \; dxdydz \; .$$

Solution:
We use spherical coordinates: note that $f(x,y,z) = x^2 + y^2 + z^2 = \rho^2$. The integral is

$$\int_0^1 \int_{\pi/4}^{3\pi/4} \int_0^{2\pi} \rho^4 \sin(\phi) \; d\theta d\phi d\rho \; .$$

There is no θ dependence in the most inner integral so that we have

$$2\pi \int_0^1 \int_{\pi/4}^{3\pi/4} \rho^4 \sin(\phi) \; d\phi d\rho = 2\pi \int_0^1 \rho^4 \int_{\pi/4}^{3\pi/4} \sin(\phi) \; d\phi d\rho \; .$$

This gives the final result $\boxed{2\pi\sqrt{2}/5}$.
Remark. The problem can also be solved in cylindrical coordinates, but it is more difficult to setup and evaluate. Here is the integral

$$\int_0^{2\pi} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_z^{\sqrt{1-z^2}} (r^2 + z^2) r \; dr dz d\theta \; .$$

For the grading, we were harsh, when the wrong region was taken. For example, if somebody would compute the correct answer for the complement which gives an anwer $2\pi/5(2-\sqrt{2})$ because the $\int\int\int_{x^2+y^2+z^2\leq 1} f \; dV = 4\pi/5$, it would lead to 5 points, if all computations were correct. (The computation of the complement with the function z^2 appeared in a practice exam.)