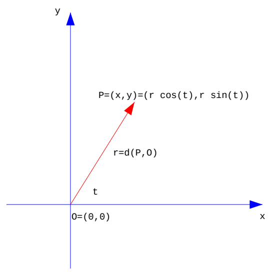
POLAR INTEGRATION **Math21a, O. Knill**

HOMEWORK: section 12.3: 4,26,36,42,44, and section 12.4: 8,20,24,28,30

POLAR COORDINATES. A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2+y^2}, \theta = \operatorname{arctg}(y/x)$. We have $x = r\cos(\theta)$, $y = r\sin(\theta)$.

Footnote: Note that $\theta = \operatorname{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ would have the same θ. In order to get the correct θ, one could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$ as Mathematica does, where $\pi/2$ is the value when $y/x = \infty$, and add π if $x < 0$ or $x = 0, y < 0$. In Mathematica, you can get the polar coordinates with $(r, \theta) = (\text{Abs}[x + Iy], \text{Arg}[x + Iy])$.



POLAR CURVES. A general polar curve is written as $(r(t), \theta(t))$. It can be translated into x, y coordinates: $x(t) = r(t)\cos(\theta(t)), y(t) = r(t)\sin(\theta(t))$.
POLAR GRAPHS. Curves which are graphs when written in polar coordinates are called **polar graphs**.

EXAMPLE. $r(\theta) = \cos(3\theta)$ is the which belongs to the class of **roses** $r(t) = \cos(nt)$.

EXAMPLE. If $y = 2x + 3$ is a line, then the equation gives $r\sin(\theta) = 2r\cos(\theta) + 3$. Solving for $r(t)$ gives $r(\theta) = 3/(\sin(\theta - 2\cos(\theta))$. The line is also a polar graph.

EXAMPLE. The polar form $r(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}$ of the ellipse (see Kepler). The ellipse is a polar graph.

INTEGRATION IN POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

$$\boxed{\int\int_R f(x,y)\ dxdy = \int\int_R g(r,\theta) r\ dr d\theta}$$

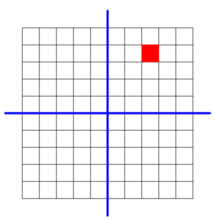
For example if $f(x, y) = x^2 + x^2 + xy$, then $g(r, \theta) = r^2 + r^2\cos(\theta)\sin(\theta)$.

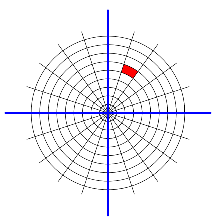
EXAMPLE. We had computed area of the disc $\{x^2 + y^2 \leq 1\}$ using substitution. Was quite a mess. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi}\int_0^1 r\ dr d\theta = 2\pi r^2/2|_0^1 = \pi\ .$$

WHERE DOES THE FACTOR "r" COME FROM?

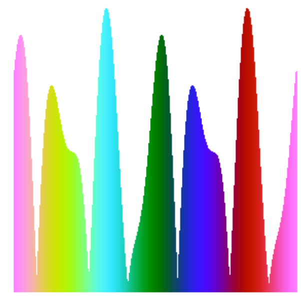
A small rectangle R with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x, y) plane. It has approximately the area $r d\theta dr$. It is small for small r.



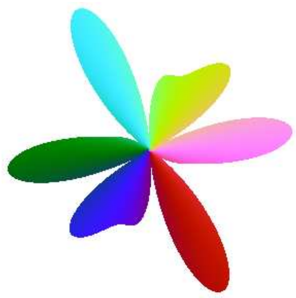


AN OTHER EXPLANATION. For fixed r, the circle $\vec{r}(\theta) = T(r, \theta) = (r\cos(\theta), r\sin(\theta))$ has the velocity vector $\vec{v} = (v_1, v_2) = (-r\sin(\theta), r\cos(\theta))$. For fixed θ, the line $\vec{r}(r) = (r\cos(\theta), r\sin(\theta))$ has the velocity vector $\vec{w} = (w_1, w_2) = (\cos(\theta), \sin(\theta))$. The parallelogram spanned by the vectors has the area $|\vec{v} \times \vec{w}| = v_1 w_2 - w_1 v_2 = r$.

ROSES. Examples of type I regions in the (θ, r) plane are **roses**: $\{(\theta, r)\ |0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



The region R in the $\theta - r$ coordinates is a type I region



The region $S = T(R)$ in the $x - y$ coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region $\{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R y\ dxdy = \int_0^{2\pi}\int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi}\frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2$$

EXAMPLE. Integrate $f(x, y) = y\sqrt{x^2 + y^2}$ over the semi annulus $R = \{(x, y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

Solution.

$$\int_1^2\int_0^\pi r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3\int_0^\pi \sin(\theta)\ d\theta dr = \frac{(2^4-1^4)}{4}\int_0^\pi \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

THE SUPER-CURVE. The Belgian Biologist Johan Gielis came up in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b}\right)^{-1/n_3}$$

It is called the **super-curve** because it can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes" (see later).

```
S[m_,n1_,n2_,n3_,a_,b_]:=Module[{},
  r1[t_]:=Abs[Cos[m t/4]/a]^n1;
  r2[t_]:=Abs[Sin[m t/4]/b]^n2;
  R[t_]:=(r1[t]+r2[t])^(1/n3);
  r[t_]:=If[R[t]==0,{0,0},{Cos[t],Sin[t]}/R[t]];
  ParametricPlot[r[t],{t,-2Pi,2Pi},PlotRange->All,
  PlotPoints->1000,Axes->False,AspectRatio->1,Frame->False]
];

RI:=Random[Integer,{3,20}]; RR:=20 Random[];
T:=S[RI,RR,RR,RR,RR,RR]
```

The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to tackle one of the more intractable problems in biology: describing form. A twist: Gielis has patented his discovery described in "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003). To the right you see the Mathematica code.