Vectors & The Dot Product

1 Are the following better described by vectors or scalars?

(a) The cost of a Super Bowl ticket.

(b) The wind at a particular point outside.

(c) The number of students at Harvard.

(d) The velocity of a car.

(e) The speed of a car.

2 Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Definition. The *dot product* $\mathbf{v} \cdot \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as follows.

- If $\mathbf{v}$ and $\mathbf{w}$ are two-dimensional vectors, say $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.
- If $\mathbf{v}$ and $\mathbf{w}$ are three-dimensional vectors, say $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

3 Compute the following dot products.

(a) $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$

(b) $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$

(c) $6\mathbf{j} \cdot 4\mathbf{k}$

(d) $\mathbf{i} \cdot (\mathbf{j} + \mathbf{k})$

Basic Properties: Here are some basic algebraic properties of the dot product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors of the same dimension and c is a scalar, then

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (c\mathbf{w})$.

4 True or false: if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors of the same dimension, then $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

5 What is the relationship between $\mathbf{v} \cdot \mathbf{v}$ and $|\mathbf{v}|$?

Main Property: Often there is an alternate definition of the dot product given. Two vectors $\mathbf{v}$ and $\mathbf{w}$ determine an angle θ in their common plane, and then their dot product is simply

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta).$$

Look back at Problems 3(cd) and 5. In each problem, what is the angle θ? Does that make sense for your answer?

6 Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

7 Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

8 True or false: If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then $|\mathbf{v} - \mathbf{w}| = |\mathbf{v}| - |\mathbf{w}|$.

9 If $\mathbf{v}$ and $\mathbf{w}$ are vectors with the property that $|\mathbf{v} + \mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$, which of the following must be true?

(a) $\mathbf{v} = \mathbf{w}$

(b) $\mathbf{v} = \mathbf{0}$

(c) $\mathbf{v}$ is orthogonal to $\mathbf{w}$

(d) $\mathbf{v}$ is parallel to $\mathbf{w}$

Vectors & The Dot Product – Answers and Solutions

1 (a) Scalar – the cost is just a number.
(b) Vector – the wind has both a speed and a direction.
(c) Scalar.
(d) Vector. The velocity is defined to be both the speed of the car (how fast it's going) and the direction it's going.
(e) Scalar. The speed refers only to how fast the car is going; it is the magnitude of the velocity vector.

2 The force Bert is applying can be described by the vector $\langle 0, 30\rangle$, while the force Ernie is applying is $\langle 40, 0\rangle$. Therefore, the resultant force is $\langle 40, 30\rangle$.

3 (a) $\langle 1, 2\rangle \cdot \langle 3, 4\rangle = 1 \cdot 3 + 2 \cdot 4 = 11$
(b) $\langle 1, 2, 3\rangle \cdot \langle 4, -5, 6\rangle = 1 \cdot 4 + 2 \cdot -5 + 3 \cdot 6 = 12$
(c) $6\mathbf{j} \cdot 4\mathbf{k} = \langle 0, 6, 0\rangle \cdot \langle 0, 0, 4\rangle = 0 \cdot 0 + 6 \cdot 0 + 0 \cdot 4 = 0$
(d) $\mathbf{i} \cdot (\mathbf{j} + \mathbf{k}) = \langle 1, 0, 0\rangle \cdot \langle 0, 1, 1\rangle = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$

4 Completely false. In fact, the statement doesn't even make sense! $\mathbf{v} \cdot \mathbf{w}$ is a scalar, and we can't dot a vector with a scalar.

5 $\mathbf{v} \cdot \mathbf{v}$ is equal to $|\mathbf{v}|^2$. Again, this is easy to see from the component definition. For a two-dimensional vector $\mathbf{v} = \langle v_1, v_2\rangle$, $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 = |\mathbf{v}|^2$. For a three-dimensional vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2 = |\mathbf{v}|^2$.

6 Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{w} = \langle 1, -1, 1\rangle$, and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then, we know that $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$. We calculate that $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 2 \cdot -1 + 1 \cdot 1 = 0$, so $0 = |\mathbf{v}||\mathbf{w}| \cos\theta$. Since the lengths $|\mathbf{v}|$ and $|\mathbf{w}|$ are both positive, $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$.

7 Let $\mathbf{v} = \langle 0, 0, 1\rangle$ and $\mathbf{w} = \langle 1, 2, 3\rangle$. We saw in class that the projection of $\mathbf{v}$ onto $\mathbf{w}$ is $\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$. In this case, $\mathbf{v} \cdot \mathbf{w} = 3$ and $\mathbf{w} \cdot \mathbf{w} = 1^2 + 2^2 + 3^2 = 14$, so the projection is $\frac{3}{14}\langle 1, 2, 3\rangle = \left\langle \frac{3}{14}, \frac{6}{14}, \frac{9}{14}\right\rangle$.

8 False. For example, let $\mathbf{v} = \langle 1, 0, 0\rangle$ and $\mathbf{w} = -\langle 1, 0, 0, \rangle$. Then, $\mathbf{v} - \mathbf{w} = \langle 2, 0, 0\rangle$, which has length 2. On the other hand, $\mathbf{v}$ and $\mathbf{w}$ both have length 1, so $|\mathbf{v}| - |\mathbf{w}| = 0$.

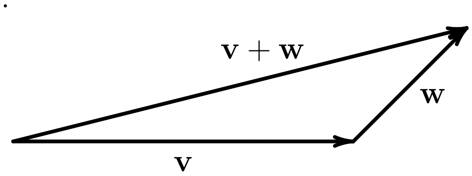
9 (c).

We can rewrite the left-hand side of the equation $|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$ using the relationship between lengths and dot products. Then, we have

$$|\mathbf{v} + \mathbf{w}|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} = |\mathbf{v}|^2 + 2\mathbf{v} \cdot \mathbf{w} + |\mathbf{w}|^2.$$

Plugging this into our original equation and cancelling like terms on both sides, we get $2\mathbf{v}\cdot\mathbf{w} = 0$ or $\mathbf{v} \cdot \mathbf{w} = 0$. This is exactly what it means for $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal.

You could also think about this problem geometrically. If $\mathbf{v}$ and $\mathbf{w}$ are not parallel, then $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ form a triangle:



The equation $|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$ says that the sides of the triangle satisfy the Pythagorean Theorem, so the triangle must be a right triangle with $\mathbf{v} + \mathbf{w}$ as the hypotenuse and $\mathbf{v}$ and $\mathbf{w}$ as the two legs. In other words, $\mathbf{v}$ and $\mathbf{w}$ must be orthogonal.