

Definition: The *cross product* of two vectors \mathbf{a} and \mathbf{b} is the vector $\mathbf{a} \times \mathbf{b}$ with

- length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$, where θ is the (smaller) angle between \mathbf{a} and \mathbf{b} , and
- direction \mathbf{n} , where \mathbf{n} is the unit vector orthogonal (perpendicular) to both \mathbf{a} and \mathbf{b} so that $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ is oriented by the right-hand rule.

1 Use this definition to compute the following cross products. I've tried to make the vectors simple so you can find $|\mathbf{a}|$, $|\mathbf{b}|$, \mathbf{n} and θ without much work.

(a) $\mathbf{i} \times \mathbf{j}$ (b) $\mathbf{i} \times (\mathbf{i} + \mathbf{j})$ (c) $\mathbf{j} \times \mathbf{i}$ (d) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

2 What is the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$? Are they the same? Are they a scalar multiple of each other (what scalar?)? Or are they not parallel at all?

3 (a) Calculate $\mathbf{i} \cdot (\mathbf{i} \times \mathbf{j})$. (You should have found $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ in Problem 1, so this asks you to compute $\mathbf{i} \cdot \mathbf{k}$.)

(b) Calculate $(\mathbf{i} + \mathbf{j}) \cdot [\mathbf{i} \times (\mathbf{i} + \mathbf{j})]$. (You should have found $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$ in Problem 1 as well, so this asks you to find $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{k}$.)

In Problem 2, you were meant to notice that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ (in fact the relationship is $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$). In Problem 3, the conclusion is that $\mathbf{a} \times \mathbf{b}$ is orthogonal (perpendicular) to both \mathbf{a} and \mathbf{b} . But that's obvious from the definition, isn't it?

More generally, we can compute a cross product in component form as follows. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then we usually write this using *determinants* (as they simplify the formulas tremendously):

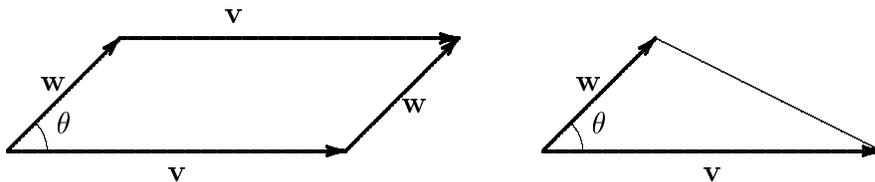
$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k} \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle. \end{aligned}$$

You have your choice: remember the determinant expression (fairly simple) or remember the last expression (ick!).

4 Practice a few cross products where it isn't straightforward to find the angle between the original vectors or the proper orthogonal vector (so the methods above won't work). Here are a few:

(a) $\langle 1, 2, 1 \rangle \times \langle 0, -1, 3 \rangle$ (b) $\langle 2, -2, 1 \rangle \times \langle 2, 1, -1 \rangle$

- 5 Any two non-zero vectors \mathbf{v} and \mathbf{w} determine a parallelogram (left) and triangle (right):



(a) What is the relationship between the area of the parallelogram and $\mathbf{v} \times \mathbf{w}$?

(b) What is the relationship between the area of the triangle and $\mathbf{v} \times \mathbf{w}$?

- 6 If the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is 0, what can you say about the vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} ?

- 7 Does the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ make sense? Pick vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} ; then compute both

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \quad \text{and} \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

(Choose wisely so that you've already done some of the work!) Did you get the same thing in both computations?

- 8 Two true-false questions:

(a) True or false: If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(b) True or false: If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then one of \mathbf{v} and \mathbf{w} is $\mathbf{0}$.

The Cross Product – Answers and Solutions

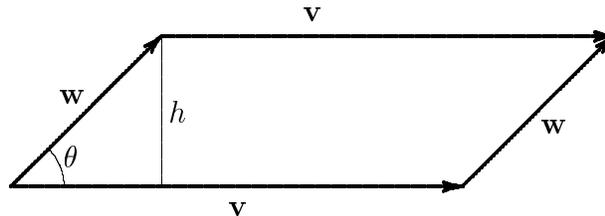
- 1 (a) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (b) $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$
 (c) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ (d) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j}) = -2\mathbf{k}$

2 $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length but opposite directions, so we have simply $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

3 Both of these (scalar) quantities are zero. One can see this either from doing the computation, or noticing simply that the cross product $\mathbf{a} \times \mathbf{b}$ is chosen to be orthogonal to both vectors \mathbf{a} and \mathbf{b} . Thus their dot product must be zero.

- 4 (a) $\langle 7, -3, -1 \rangle$ (b) $\langle 1, 4, 6 \rangle$

5 (a) The area of the parallelogram is $|\mathbf{v} \times \mathbf{w}|$. Here's a picture that might help explain:



By simple trigonometry, we have that the height is $h = |\mathbf{w}| \sin(\theta)$, so the area of the parallelogram is (the base times the height) $|\mathbf{v}|h = |\mathbf{v}| |\mathbf{w}| \sin(\theta)$, or $|\mathbf{v} \times \mathbf{w}|$.

(b) The area of the triangle is $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$ as the triangle is precisely half of the parallelogram of part (a).

6 This means the \mathbf{u} , \mathbf{v} , and \mathbf{w} are coplanar. The dot product vanishing implies that \mathbf{w} is perpendicular to $\mathbf{u} \times \mathbf{v}$. By the definition of the cross product, both \mathbf{u} and \mathbf{v} are perpendicular to $\mathbf{u} \times \mathbf{v}$ as well. Thus \mathbf{u} , \mathbf{v} , and \mathbf{w} all have a common perpendicular, so they all lie on the same plane.

7 The expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ doesn't make sense, since the two expressions

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \quad \text{and} \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

can be different than each other. For example, if $\mathbf{u} = \mathbf{v} = \mathbf{i}$ and $\mathbf{w} = \mathbf{j}$, then

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \quad \text{but} \quad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

8 (a) This is false. For example, $\mathbf{u} \times (\mathbf{u} + \mathbf{v}) = \mathbf{u} \times \mathbf{v}$ for any \mathbf{u} and \mathbf{v} . This was done in Problem 1(a): $\mathbf{i} \times \mathbf{j} = \mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$.

(b) This is true. An easy way to see it is to write the two equations as $|\mathbf{u}| |\mathbf{v}| \sin(\theta) = 0$ and $|\mathbf{u}| |\mathbf{v}| \cos(\theta) = 0$. We can't have both $\sin(\theta) = 0$ and $\cos(\theta) = 0$, so we must have either $|\mathbf{u}| = 0$ or $|\mathbf{v}| = 0$. Hence one of \mathbf{u} or \mathbf{v} is the zero vector.