Lines and Planes

Lines in Space How can we express the equation(s) of a line through a point (x_0, y_0, z_0) and parallel to the vector $\mathbf{u} = \langle a, b, c\rangle$? Many ways: as *parametric (scalar) equations:*

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc;$$

as a *parametric vector equation:*

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u} \qquad (\text{where } \mathbf{r} = \langle x, y, z\rangle \text{ and } \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle);$$

or by *symmetric* equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

1 Let L be the line which passes through the points $(1, -2, 3)$ and $(4, -4, 6)$.

(a) Find a parametric vector equation for L.

(b) Find parametric (scalar) equations for L.

(c) Find symmetric equations for L.

2 How could we write symmetric equations for a line with, say, $c = 0$? Try this for the line through the points $(5, 2, 2)$ and $(3, -1, 2)$.

Planes in Space The equation of a plane through the point (x_0, y_0, z_0) and perpendicular (or normal or orthogonal) to the vector $\mathbf{n} = \langle a, b, c\rangle$ also has many (equivalent) equations:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

(where again $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$); or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{or} \qquad ax + by + cz + d = 0$$

(where d is a constant).

3 Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3\rangle$.

4 Find an equation describing the plane which passes through the points $P(2,2,1)$, $Q(3,1,0)$, and $R(0,-2,1)$.

5 Let L_1 be the line with parametric vector equation $\mathbf{r}_1(t) = \langle 7,1,3\rangle + t\langle 1,0,-1\rangle$ and L_2 be the line described parametrically by $x = 5, y = 1+3t,\ z = t$. How many planes are there which contain L_2 and are parallel to L_1? Find an equation describing one such plane.

Distances Between Points, Lines, and Planes

6 Find the distance from the point $(0,1,1)$ to the plane $2x+3y+4z = 15$.

7 Find the distance from the point $(1,3,-2)$ to the line $\frac{x}{3} = y-1 = z+2$.

8 Two true-false questions:

(a) True or false: The line $x = 2t,\ y = 1+3t,\ z = 2+4t$ is parallel to the plane $x-2y+z = 7$.

(b) True or false: Let S be a plane normal to the vector $\mathbf{n}$, and let P and Q be points not on the plane S. If $\mathbf{n}\cdot\overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.

Lines and Planes – Answers and Solutions

1 (a) $\mathbf{r}(t) = \langle 1, -2, 3\rangle + t\langle 3, -2, 3\rangle$

(b) $x = 1 + 3t,\ y = -2 - 2t,\ z = 3 + 3t$

(c) $\dfrac{x-1}{3} = \dfrac{y+2}{-2} = \dfrac{z-3}{3}$

2 The parameterization involves simply $z = 2$, so the symmetric equation reduces to

$$\frac{x-5}{-2} = \frac{y-2}{-3} \qquad z = 2.$$

3 Many possibilities:

$$\begin{aligned}
&\langle 2, 1, -3\rangle \cdot \langle x-1, y-3, z-5\rangle = 0\\
&\langle 2, 1, -3\rangle \cdot \langle x, y, z\rangle = \langle 2, 1, -3\rangle \cdot \langle 1, 3, 5\rangle\\
&2(x-1) + (y-3) - 3(z-5) = 0\\
&2x + y - 3z + 10 = 0
\end{aligned}$$

Any one is fine.

4 Here we need an extra step to find the normal $\mathbf{n}$. We find this by finding two vectors in the plane and computing their cross product. We will write $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$, so

$$\begin{aligned}
\mathbf{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, -1, -1\rangle \times \langle -2, -4, 0\rangle\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & -4 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ -2 & -4 \end{vmatrix}\mathbf{k}\\
&= \langle -4, 2, -6\rangle.
\end{aligned}$$

Thus the plane is

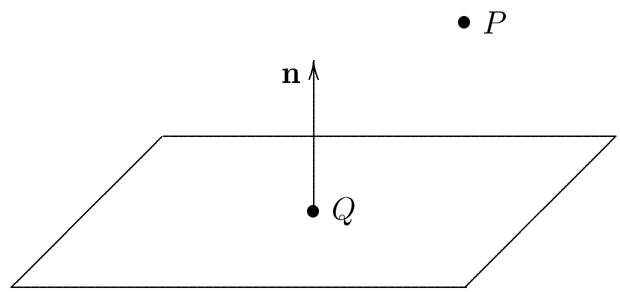
$$\langle -4, 2, -6\rangle \cdot \langle x-2, y-2, z-1\rangle = 0 \qquad \text{or} \qquad -4x + 2y - 6z + 10 = 0.$$

5 Observe that L_1 goes through the point $(7, 1, 3)$ and is parallel to the vector $\langle 1, 0, -1\rangle$ while L_2 goes through $(5, 1, 0)$ and is parallel to the vector $\langle 0, 3, 1\rangle$.

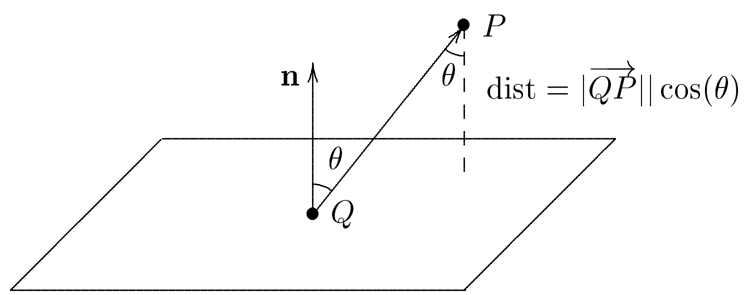
Since L_1 and L_2 are not parallel (which we can see because the vectors $\langle 1, 0, -1\rangle$ and $\langle 0, 3, 1\rangle$ are not parallel), there is only one such plane.

Therefore, the plane in question must be parallel to both $\langle 1, 0, -1\rangle$ and $\langle 0, 3, 1\rangle$, so it is orthogonal to $\langle 1, 0, -1\rangle \times \langle 0, 3, 1\rangle = \langle 3, -1, 3\rangle$. That is, $\mathbf{n} = \langle 3, -1, 3\rangle$ is a normal vector for the plane. In addition, the plane goes through $(5, 1, 0)$. So, the plane has equation $\langle 3, -1, 3\rangle \cdot \langle x-5, y-1, z\rangle = 0$ or $3x - y + 3z - 14 = 0$.

6 We have a point $P(0,1,1)$ and a plane, and we want to find the distance between the two. Here is one method. Suppose Q is *any* point in the plane and $\mathbf{n}$ is a normal vector for the plane.

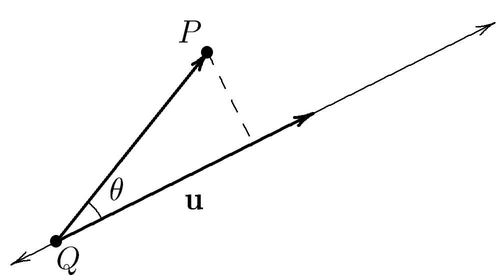


Then the distance from P to the plane is simply the (absolute value of the) scalar projection $|\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP}| = |\overrightarrow{QP}||\cos(\theta)| = \left|\frac{|\mathbf{n}\cdot\overrightarrow{QP}|}{|\mathbf{n}|}\right|$:

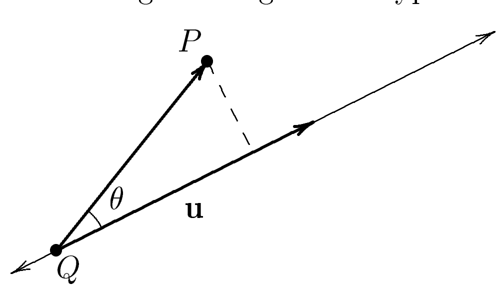


In this particular problem, we can take $Q = (0,5,0)$ as our point on the plane, and $\mathbf{n} = \langle 2,3,4\rangle$ is the normal vector for the plane. Then $\overrightarrow{QP} = \langle 0,-4,1\rangle$ and $\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP} = \frac{\mathbf{n}\cdot\overrightarrow{QP}}{|\mathbf{n}|} = \frac{-8}{\sqrt{29}}$. Thus the distance is $\frac{8}{\sqrt{29}}$.

7 We have a point $P(1,3,-2)$ and a line, and we want to find the distance between the two:



Here's one way to do that. Find a point Q on the line and a vector $\mathbf{u}$ parallel to the line. The distance is then the height of of the right triangle with hypotenuse $\overrightarrow{QP}$ and base on the line:



This height is simply $|\overrightarrow{QP}|\sin(\theta)$, which we recognize as most of the length of the cross product of $\overrightarrow{QP}$ with $\vec{u}$: $|\overrightarrow{QP}|\sin(\theta) = \frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$.

For the given line, it will be easier to find a point on the line and a vector parallel to the line if we rewrite it using a parametric vector equation. To do this, let's set t equal to $\frac{x}{3} = y - 1 = z + 2$. Then, $x = 3t$, $y = 1 + t$, and $z = -2 + t$, so we can describe the line by the parametric vector equation $\langle 0, 1, -2\rangle + t\langle 3, 1, 1\rangle$. From this, we can see that $Q(0, 1, -2)$ is a point on the line and $\mathbf{u} = \langle 3, 1, 1\rangle$ is a vector parallel to the line.

Now, we just compute $\frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$: $\overrightarrow{QP} = \langle 1, 2, 0\rangle$, so $\overrightarrow{QP}\times\mathbf{u} = \langle 2, -1, -5\rangle$ and $\dfrac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|} = \sqrt{\dfrac{30}{11}}$.

8 (a) True.

A normal vector for the plane is $\mathbf{n} = \langle 1, -2, 1\rangle$. The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the vector $\langle 2, 3, 4\rangle$, and this vector is orthogonal to $\mathbf{n}$, so this vector must be parallel to the plane.

Another way to see that the line and plane are parallel is to try to compute the intersection. If (x, y, z) is in both the line and plane, then the four equations $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$, and $x - 2y + z = 7$ must all be satisfied. Plugging the first three equations into the fourth, $2t - 2(1 + 3t) + (2 + 4t) = 7$, which simplifies to $0 = 7$, so there are no solutions to all four equations. This means that the line and plane do not intersect, so they must be parallel.

(b) True. The fact that $\mathbf{n}\cdot\overrightarrow{PQ} = 0$ means that $\mathbf{n}$ is orthogonal to $\overrightarrow{PQ}$, so $\overrightarrow{PQ}$ is parallel to the plane.