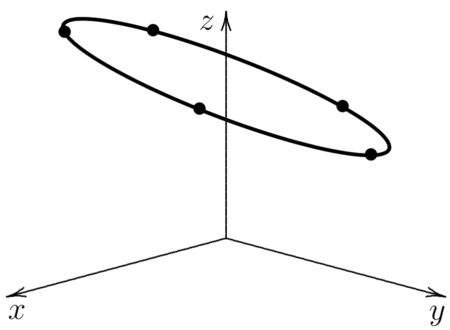
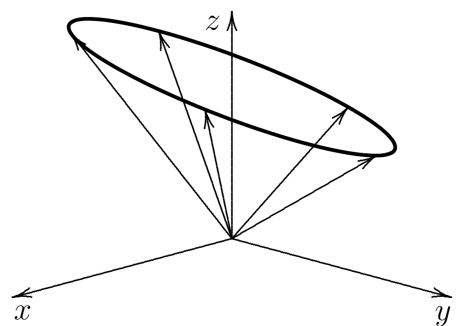
Vector-Valued Functions

We're now going to study *vector-valued functions* or *parameterized curves*. We can write this as either

$$x = f(t) \quad y = g(t) \quad z = h(t) \qquad \text{or} \qquad \mathbf{r}(t) = \langle x, y, z \rangle = \langle f(t), g(t), h(t) \rangle.$$

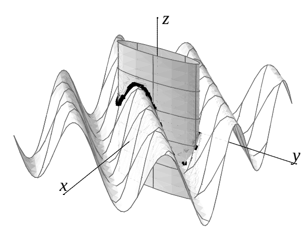
The difference is as illustrated in the two pictures below. Both are the same curve, but written as a parameterized curve (on the left, with the equations from the left) it is a collection of points while as a vector-valued function the curve is the trace of the heads of a collection of position vectors (vectors with their feet at the origin).



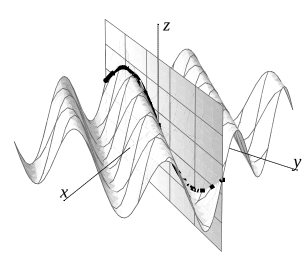


1. Problem 1 is the matching problem on the back of this page. Start there!

2. (a) The surfaces $9x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.



(b) The surfaces $z = \sin(x - y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.



1 Match each vector-valued function to the curve it parameterizes.

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

(b) $\mathbf{r}(t) = \langle t\cos t, t\sin t, t \rangle$

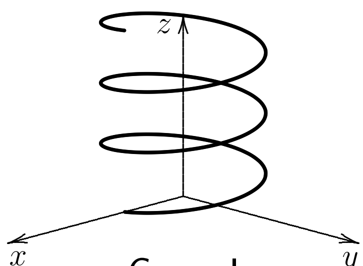
(c) $\mathbf{r}(t) = \langle \cos t, \sin t, t^3 \rangle$

(d) $\mathbf{r}(t) = \langle \cos t^3, \sin t^3, t^3 \rangle$

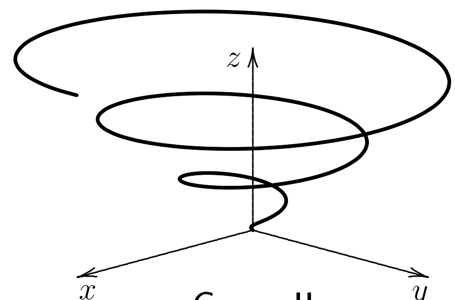
(e) $\mathbf{r}(u) = \langle \cos u, \sin u, 1 + \sin 4u \rangle$

(f) $\mathbf{r}(u) = \langle \cos u, \sin u, 1 + 4\sin u \rangle$

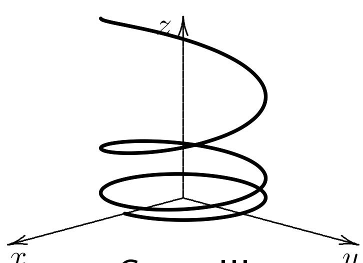
(g) $\mathbf{r}(t) = \langle 2\cos t, 1 + 4\cos t, 3\cos t \rangle$



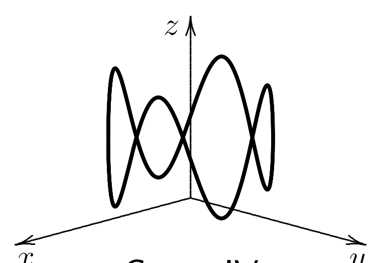
Curve I



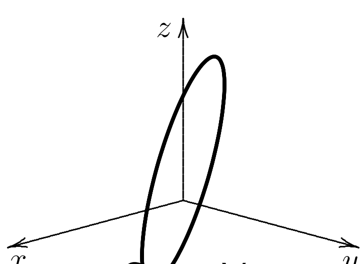
Curve II



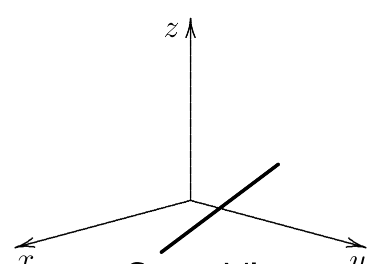
Curve III



Curve IV



Curve V



Curve VI

Vector-Valued Functions – Answers and Solutions

1 (a) This is Curve II, the *helix*. When looked on from above (from the positive z direction), this curve is simply a circle in the xy-plane. The $z = t$ component lifts the circle into the helix spinning above the circle in the plane.

If we visualize this as a particle at the tip of the position vector $\mathbf{r}(t)$, then from above it looks like the particle is simply spinning in a circle. But we also know that $z = t$, so the particle is rising at a constant rate. Hence the helix.

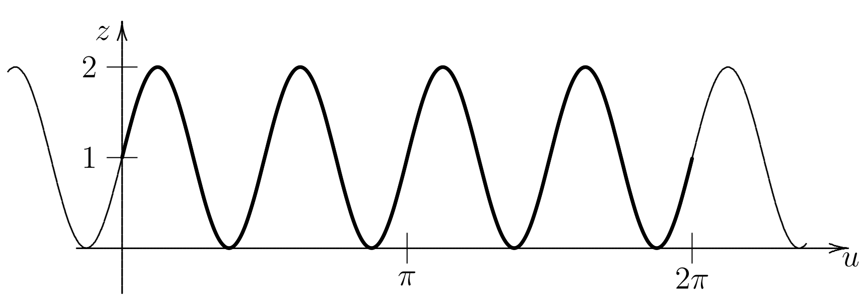
(b) This is Curve II. This is very similar to part (a), except now the x and y components have a changing magnitude (namely, $|t|$). Projected onto the xy-plane (or viewed from above), this is a spiral $\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle$, and in space it is this elongated spiral.

(c) This is Curve III. This is again very similar to part (a), except now the z coordinate rises very slowly at first (so the lower rings are close together), then very quickly (so the higher rings are farther apart).

(d) This is Curve I again. If we make the substitution $s = t^3$, then the curve is $\mathbf{r}(s) = \langle \cos s, \sin s, s\rangle$ (the same as part (a), although with a differently named parameter). If we visualize this vector-valued function as telling us not only the path we traverse, but also how quickly we travel along it, we find the difference between (a) and (d). In part (a), we rise at a constant speed (since $z = t$), but in part (d) we rise slowly at first then very quickly (here $z = t^3$).

We trace out exactly the same curve in parts (a) and (d), but the rate at which we trace out the curve differs between (a) and (d). Put another way, the particle (of the answer to part (a)) travels at a constant rate in part (a) but a varying (slower at first, then faster and faster) rate in part (d).

(e) This is Curve IV. Again as we look from above we have a unit circle in the xy-plane. But what happens to the height z as u passes from $u = 0$ to $u = 2\pi$ (through one circle in the xy-plane)? Here's a graph of $z = 1 + \sin(4u)$ in the uz-plane, with the graph over $[0, 2\pi]$ highlighted:



Thus we can see that when our particle makes one orbit in the xy-plane, the height z rises and falls a total of four cycles. Thus we're looking at Curve IV.

(f) This is Curve V. We can make a similar argument here as we did in part (e), except now z makes one cycle (instead of four) for every cycle that x and y make. Another approach is to consider the z coordinates as bounded between -3 and 5 while x and y coordinates proceed in a circle. Both these approaches point to Curve V.

Another interesting approach is to notice that $y = \sin u$ and $z = 1+4\sin u$, so $z = 1+4y$. Thus our curve must lie on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 1 + 4y$. In fact our curve is the ellipse of intersection of these two surfaces.

(g) This is Curve VI. If we make the substitution $u = \cos t$, we get

$$\mathbf{r} = \langle 2u, 1 + 4u, 3u\rangle = \langle 0, 1, 0\rangle + u\langle 2, 4u, u\rangle.$$

This is the equation of a line.

We should note that $-1 \leq u \leq 1$ (since $u = \cos t$), so we actually have only a segment of the line. Thus Curve VI is our line segment.

2 (a) Notice that we already know how to express z in terms of x and y because $z = \sin(x-y)$. Therefore, if we can express both x and y in terms of a parameter t, we will automatically be able to express z in terms of that parameter as well.

So, let's focus on the relationship between x and y, which is given by the equation $9x^2 + \frac{y^2}{4} = 1$. If we rewrite this as $(3x)^2 + (y/2)^2 = 1$, then we see that we can write $3x = \cos t$, $y/2 = \sin t$, or $x = \frac{1}{3}\cos t$ and $y = 2\sin t$.

Since $z = \sin(x - y)$, we now have $z = \sin\left(\frac{1}{3}\cos t - 2\sin t\right)$. We can also write this as the vector-valued function

$$\mathbf{r}(t) = \left\langle \tfrac{1}{3}\cos t, 2\sin t, \sin(\tfrac{1}{3}\cos t - 2\sin t)\right\rangle.$$

(b) As in the previous part, it's easy to express z in terms of x and y, so we should focus on writing x and y in terms of a parameter t. Notice, however, that this time it's also easy to write y in terms of x, since $y = 2x$. Therefore, we can simply let x be the parameter, $x = t$. Then, $y = 2x = 2t$, and $z = \sin(x - y) = \sin(t - 2t) = \sin(-t) = -\sin t$. Written as a vector-valued function, $\mathbf{r}(t) = \langle t, 2t, -\sin t\rangle$.