

For each of the following surfaces,

- (a) write the surface in parameterized form as $\mathbf{r}(u, v)$, and
(b) describe the “grid curves” $u = k$ and $v = k$ for your parameterization.

1 $3x + 2y + z = 6$

2 $x^2 + y^2 + z^2 = 4$ but $z \geq 0$

3 $\frac{x^2}{4} - \frac{y^2}{9} + z^2 = 1$

4 The graph of the curve
 $y = 1 + \sin(x)$ revolved
around the x -axis

5 $-\frac{x^2}{9} - \frac{y^2}{16} + z^2 = 1$

6 The graph of the curve
 $y = x^2 + x$ ($0 \leq x \leq 2$)
revolved around the x -axis

Identify each of the following parameterized surfaces:

7 $\mathbf{r}(u, v) = \langle 2 + u, 3 - u - v, v + 5 \rangle$

8 $\mathbf{r}(u, v) = \langle u, \cos(v), \sin(v) \rangle$

9 $\mathbf{r}(u, v) = \langle u, v, v^2 - u^2 \rangle$

10 $\mathbf{r}(u, v) = \langle u^2, 4 \cos(v), 5 \sin(v) \rangle$

11 $\mathbf{r}(u, v) = \langle u, u \sin(v), u \cos(v) \rangle$

12 $\mathbf{r}(u, v) = \langle u \cos(v), \sqrt[3]{u}, u \sin(v) \rangle$

On this page, match the equation of the surface to the appropriate graph. Give reasons!

(a) $\mathbf{r}(u, v) = \langle u, \sin(u) \cos(v), \sin(u) \sin(v) \rangle$

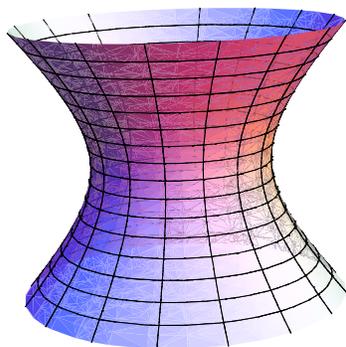
(c) $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2} \mathbf{k}$

(e) $\mathbf{r}(u, v) = \langle av \cos(u), bv \sin(u), 2v^2 \rangle$

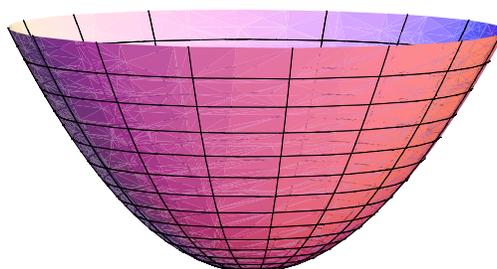
(b) $\mathbf{r}(u, v) = \langle u + 1, v - 2, 3 - u - 2v \rangle$

(d) $\mathbf{r}(u, v) = \langle \sin(u) \cos(v), \cos(u), \sin(u) \sin(v) \rangle$
 $0 \leq u \leq \pi/2, 0 \leq v \leq \pi$

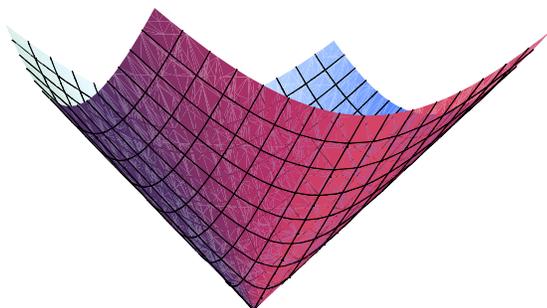
(f) $\mathbf{r}(u, v) = \langle \sqrt{1 + u^2} \cos(v), \sqrt{1 + u^2} \sin(v), u \rangle$



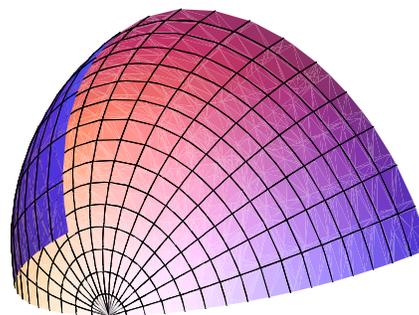
Graph I



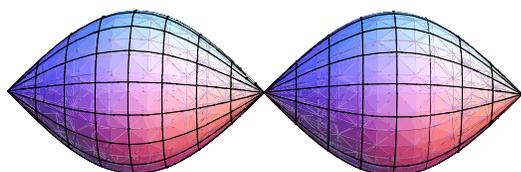
Graph II



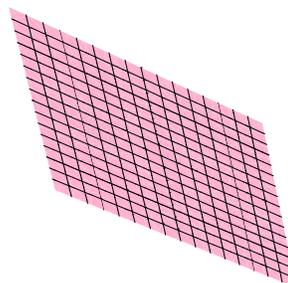
Graph III



Graph IV



Graph V



Graph VI

Parametric Surfaces – Answers and Solutions

- 1 (a) One choice is $\mathbf{r}(u, v) = \langle u, v, 6 - 3u - 2v \rangle$. Many many others are possible.
 (b) When $u = k$ we get the line $\mathbf{r}(v) = \langle k, v, 6 - 3k - 2v \rangle$. For example, $\mathbf{r}(v) = \langle 1, v, 3 - 2v \rangle$ or $\mathbf{r} = \langle 1, 0, 3 \rangle + v\langle 0, 1, -2 \rangle$ when $k = 1$. Similarly, we get the line $\mathbf{r}(u) = \langle 0, k, 6 - 2k \rangle + u\langle 1, 0, -3 \rangle$ when $v = k$.

- 2 (a) Two reasonable choices are $\mathbf{r}(u, v) = \langle u, v, \sqrt{4 - u^2 - v^2} \rangle$ or (using spherical coordinates):

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin(\phi) \cos(\theta), 2 \sin(\phi) \sin(\theta), 2 \cos(\phi) \rangle \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta < 2\pi.$$

- (b) The grid curves for the first parameterization are the lines on the hemisphere directly above the lines $x = k$ and $y = k$. In the spherical coordinate parameterization, the grid curves are latitude lines (circles at a fixed height) for $\phi = k$ and arcs of great circles from the “north pole” to the “equator” for $\theta = k$.

- 3 (a) Here we agreed to assume that $y \geq 0$ (although see (f) on page 2 for a similar example with a single parameterization). We could start out $x = u$ and $z = v$, but instead we use the parameterization

$$\mathbf{r}(u, v) = \langle 2v \cos(u), 3\sqrt{v^2 - 1}, v \sin(u) \rangle \quad 0 \leq u < 2\pi, \quad v \geq 1.$$

- (b) The grid curves corresponding to $v = k$ are (provided $k \geq 1$) ellipses in the $y = 3\sqrt{k^2 - 1}$ plane, parallel to the xz -plane. The $u = k$ grid curves are hyperbolas in space over the ray corresponding to the angle $\theta = u$ in the xy -plane.

- 4 (a) The usual parameterization for this surface of revolution is

$$\mathbf{r}(u, v) = \langle u, (1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v) \rangle, \quad 0 \leq v < 2\pi.$$

This is simply $\mathbf{r}(x, \theta) = \langle x, f(x) \cos(\theta), f(x) \sin(\theta) \rangle$.

- (b) The $u = k$ grid curves are circles lying in planes parallel to the yz -plane, while the $v = k$ grid curves are copies of the graph of $y = 1 + \sin(x)$ rotated through an angle of k about the x -axis.

- 5 This is a double-sheeted hyperboloid, so we will restrict ourselves to only one sheet. Let's assume that $z \geq 0$.

- (a) One way to do this is to simply use the $x = u, y = v$ parameterization, from which we find that

$$\mathbf{r}(u, v) = \left\langle u, v, \sqrt{1 + \frac{u^2}{9} + \frac{v^2}{16}} \right\rangle.$$

Another approach is to notice that the $z = k$ trace is an ellipse and to start with that ellipse. Let's let $x = 3u \cos(v)$ and $y = 4u \sin(v)$, so $z^2 = 1 + \frac{(3u \cos(v))^2}{9} + \frac{(4u \sin(v))^2}{16} = 1 + u^2$. Thus our parameterization becomes

$$\mathbf{r}(u, v) = \left\langle 3u \cos(v), 4u \sin(v), \sqrt{1 + u^2} \right\rangle.$$

We'll use this one in part b.

(b) When $u = k$, our grid curve is an ellipse in the plane $z = \sqrt{1 + k^2}$. When $v = k$, we again (as in Problem 3) have a portion of a hyperbola lying over the ray with $\theta = k$ in the xy -plane.

6 (a) Following the example of Problem 4, we have as our parameterization

$$\mathbf{r}(u, v) = \langle u, (u^2 + u) \cos(v), (u^2 + u) \sin(v) \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v < 2\pi.$$

(b) The $u = k$ grid curves are circles in planes parallel to the yz -plane, while the $v = k$ grid curves are copies of the graph $y = x^2 - x$ rotated through an angle $\theta = k$ around the x -axis. Note the similarity of this answer to the answer in Problem 4(b).

7 This is a plane. Solve the equations

$$x = 2 + u \qquad y = 3 - u - v \qquad z = v + 5$$

for u and v . We get $u = x - 2$ and $v = z - 5$. Plugging these into the equation for y , we get $y = 3 - (x - 2) - (z - 5) = 10 - x - z$, or $x + y + z = 10$.

8 This is a cylinder. The traces $u = k$ are circles in planes parallel to the yz -plane, whereas the the $v = k$ traces are lines parallel to the x -axis through the point $(0, \cos(k), \sin(k))$.

9 This is the graph of $z = y^2 - x^2$. This is a hyperbolic paraboloid (see the table on page 682 of Stewart).

10 This is very similar to Problem 8. The differences: we now have ellipses (rather than circles) as the $u = k$ traces parallel to the yz -plane, and the $v = k$ traces are only rays, not lines (since $x = u^2 \geq 0$). Thus we get what might be called an elliptical half-cylinder.

11 The $u = k$ traces here circles of radius k centered on the x -axis in the plane $x = k$. This is a cone. (The $v = k$ traces are lines that project line with an angle $\theta = v$ in the yz -plane. These lines make a 45° angle with the x -axis.

12 The $u = k$ traces are circles of radius k centered on the y -axis in the plane $y = \sqrt[3]{k}$. The $v = k$ traces are curves that are actually rotations of the $v = 0$ curve ($\mathbf{r}(u) = \langle u, \sqrt[3]{u}, 0 \rangle$ is the $v = 0$ curve) about the y -axis. In fact we have a surface of revolution.

(If we make the change $t = \sqrt[3]{u}$, we get the parameterization $\mathbf{r}(t, v) = \langle t^3 \cos(v), t, t^3 \sin(v) \rangle$. This looks more like the surface of revolution example from the text, except the axis of revolution here is the y -axis, not the x -axis.

Matching – Answers and Solutions

The answers are:

(a) Graph V	Graph I: (f)
(b) Graph VI	Graph II: (e)
(c) Graph III	Graph III: (c)
(d) Graph IV	Graph IV: (d)
(e) Graph II	Graph V: (a)
(f) Graph I	Graph VI: (b)

Here are some explanations:

- (a) This is a surface of revolution, rotating $y = \sin(x)$ around the x -axis. Thus Graph V.
- (b) This is a plane, similar to Problem 7. Hence Graph VI.
- (c) This is Graph III, the half-cone. One way to see this is to stare at $\mathbf{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2} \rangle$ and recognize that this means $x = u$, $y = v$, and $z = \sqrt{u^2 + v^2} = \sqrt{x^2 + y^2}$. Thus $z^2 = x^2 + y^2$, or (in cylindrical coordinates) $z = r$. This is a cone. Since our z coordinate is always non-negative, we have only the top half of the cone. Thus Graph III.
- (d) This parameterization is simply spherical coordinates with $\rho = 1$, $u = \phi$, and $v = \theta$ (and x , y , and z permuted). The restrictions on u and v mean that it's only a quarter of the sphere of radius 1, or Graph IV.
- (e) Here the $v = k$ grid curves are ellipses at a height $z = 2k^2$. The $u = k$ grid traces are half-parabolas over the ray in the xy -plane at angle $\theta = k$. Another approach is to notice that this curve satisfies the equation
- $$\frac{z}{2} = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$
- which is an elliptic paraboloid. Both approaches point to Graph II.
- (f) This is another surface of revolution. We're revolving $x = \sqrt{1 + z^2}$ around the z -axis. This curve is a hyperbola, so grid curves are either hyperbolas or circles. Thus we end up with the one-sheeted hyperboloid shown in Graph I.