Partial Derivatives

Define the partial derivative of $f(x,y)$ with respect to x by

$$f_x = \frac{\partial}{\partial x}(f) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and similarly the partial derivative of $f(x,y)$ with respect to y by

$$f_y = \frac{\partial}{\partial y}(f) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

For each of the following functions, compute both first partial derivatives f_x and f_y (or f_t):

1 $f(x,y) = e^x \cos(y)$

2 $f(x,y) = x^3 - 3xy^2$

3 $f(x,t) = e^{-(x+t)^2}$

4 $f(x,t) = \sin(x-t) + \sin(x+t)$

We can compute higher order derivatives by simply repeating the process. For example,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}.$$

Compute the four second partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} for the four functions above.

5 These four functions above were selected because they solve some *partial differential equations* or PDEs. Listed below are four common PDEs, some of which you will see in the homework. Determine which function is a solution of which PDE by substituting in derivatives.

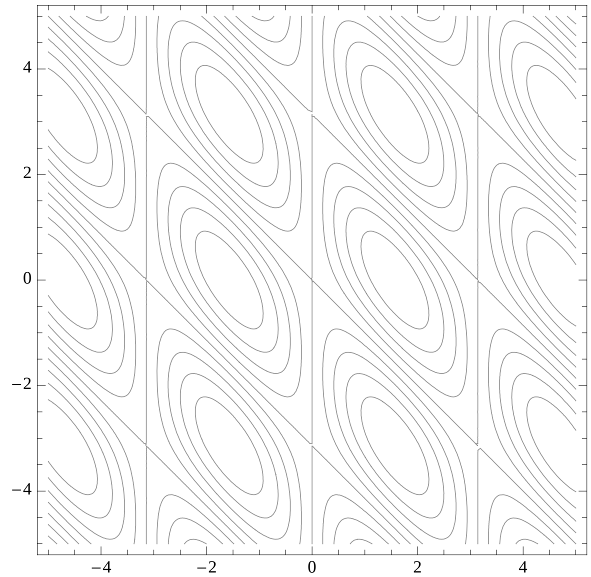
Laplace Equation: $f_{xx} + f_{yy} = 0$

Advection (Transport) Equation: $f_t = f_x$

Wave Equation: $f_{tt} = f_{xx}$

Heat Equation: $f_t = f_{xx}$

6 Here is a contour plot for the function $f(x, y) = \sin(x)\sin(x + y)$.



Without actually computing the derivatives, answer the following questions:

(a) What is the sign of f_x at $(x, y) = (1, 0)$?

(b) What is the sign of f_y at $(x, y) = (1, 0)$?

(c) What is the sign of f_{xx} at $(x, y) = (\frac{1}{2}, 1)$?

(d) What is the sign of f_{xy} at $(x, y) = (\frac{1}{2}, 1)$?

7 You may notice that $f_{xy} = f_{yx}$ in all of the above cases. This is a consequence of *Clairaut's Theorem.*

(a) Use Clairaut's Theorem to compute the requested derivatives of the following functions:

(i) f_{xyxyxy} if $f(x, y) = x^2 \cos(e^y + y^2)$

(ii) f_{xxxyy} if $f(x, y) = x^3y^2 - \frac{y}{x+\ln(x)}$

Partial Derivatives – Answers and Solutions

1 For $f(x,y) = e^x \cos(y)$, we get first derivatives

$$f_x = \frac{\partial f}{\partial x} = e^x \cos(y) \qquad \text{and} \qquad f_y = \frac{\partial f}{\partial y} = -e^x \sin(y).$$

The second derivatives are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^x \cos(y), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -e^x \cos(y),$$

and

$$f_{xy} = f_{yx} = -e^x \sin(y) \qquad \left(\text{where } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } f_{yx} = \frac{\partial^2 f}{\partial x \partial y}\right).$$

Notice that $f_{xx} + f_{yy} = e^x \cos(y) - e^x \cos(y) = 0$, so this function $f(x,y)$ satisfies the Laplace equation.

2 For $f(x,y) = x^3 - 3xy^2$, we get first derivatives

$$f_x = 3x^2 - 3y^2 \qquad \text{and} \qquad f_y = -6xy$$

and second derivatives

$$f_{xx} = 6x, \qquad f_{xy} = f_{yx} = -6y, \qquad \text{and} \qquad f_{yy} = -6x.$$

Notice again that $f_{xx} + f_{yy} = 6x - 6x = 0$, so this $f(x,y)$ is a solution to the Laplace equation.

3 For $f(x,t) = e^{-(x+t)^2}$, we get first derivatives

$$f_x = f_t = -2(x+t)e^{-(x+t)^2}$$

and second derivatives

$$f_{xx} = f_{xt} = f_{tx} = t_{tt} = \left(4(x+t)^2 - 2\right) e^{-(x+t)^2}.$$

Notice that $f_t = f_x$ and $f_{tt} = f_{xx}$, so this $f(x,y)$ is a solution to both the advection equation and the wave equation.

4 For $f(x,t) = \sin(x-t) + \sin(x+t)$, we get first derivatives

$$f_x = \cos(x-t) + \cos(x+t) \qquad \text{and} \qquad f_t = -\cos(x-t) + \cos(x+t)$$

and second derivatives

$$f_{xx} = -\sin(x-t) - \sin(x+t), \qquad f_{xt} = f_{tx} = \sin(x-t) - \sin(x+t),$$

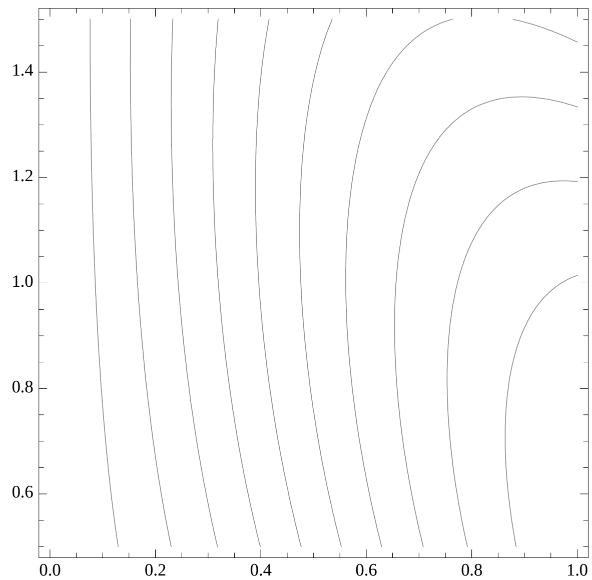
and

$$f_{xx} = -\sin(x-t) - \sin(x+t).$$

Notice again that $f_{tt} = f_{xx}$, so this $f(x,y)$ is also a solution to the wave equation.

6 (a) and (b): Both $f_x > 0$ and $f_y > 0$ at $(x, y) = (1, 0)$. The idea is simply that there is a maximum at $(x, y) = (\frac{\pi}{2}, 0) \approx (1.57, 0)$ (where $f(\frac{\pi}{2}, 0) = 1$), which is in the circled region on the contour plot. Since the the vertical lines $x = k\pi$ and diagonal lines $y = -x + k\pi$ are the only places where $f(x, y) = 0$, we see that the function is increasing in both the positive x and positive y directions at $(1, 0)$.

(c) Here is a "zoomed-in" contour plot near the point $(x, y) = (\frac{1}{2}, 1)$:



As above, $f_x > 0$ at this point. As we move to the right, however, the contour lines become spaced farther apart. Thus f is increasing slower, so f_x is decreasing (although still positive). Thus $f_{xx} < 0$.

(d) Looking at the same picture as in part (c), we see that the contour lines again become spaced farther apart (horizontally) as we move up vertically. Thus, again, $f_{xy} < 0$ at $(x, y) = (\frac{1}{2}, 1)$.

7 The point of both these problems is to re-order the derivatives so that you take the "easier" derivatives first.

(a) Here we take the x derivatives first:

$$\begin{aligned} f &= x^2 \cos\left(e^y + y^2\right) \\ f_x &= 2x \cos\left(e^y + y^2\right) \\ f_{xx} &= 2 \cos\left(e^y + y^2\right) \\ f_{xxx} &= 0; \end{aligned}$$

so $f_{xyxyxy} = f_{xxxyyy} = 0$.

(b) Here we take the y derivatives first. We get

$$\begin{aligned} f &= x^3y^2 - \frac{y}{x+\ln(x)} \\ f_y &= 2x^3y - \frac{1}{x+\ln(x)} \\ f_{yy} &= 2x^3, \end{aligned}$$

and so (after more derivatives) $f_{yyxxx} = 2 \cdot 3 \cdot 2 = 12$. Thus $f_{xxxyy} = 12$ as well.