Approximately Tangent Planes

1 To find the *tangent plane* to the graph $z = f(x, y)$ of a surface at the point $(x, y) = (a, b)$ (really $(x, y, z) = (a, b, f(a, b))$), we begin by finding vectors tangent to the surface at this point.

(a) The x grid curve (or $y = a$ trace) through this point is parameterized as $\mathbf{r}_1(x) = \langle x, b, f(x, b)\rangle$. Find the tangent vector to this curve at $x = a$. (This is a tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$.)

(b) Find another tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$ using the y grid curve $(x = a)$.

(c) Find the normal vector to the tangent plane using the two tangent vectors from parts (a) and (b).

(d) Now find the equation of the tangent plane. This is the plane through $(a, b, f(a, b))$ and perpendicular to the normal vector from part (c).

2 Mostly the tangent plane is used for approximation. Here is a simple example.

(a) Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2$ at the point $(x, y, z) = (1, 1, 3)$.

(b) Your answer to part (a) (and part (d) of Problem 1) should involve the formula

$$z \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Use this to approximate the values of $z = f(x, y) = x^2 + 2y^2$ at the points $(x, y) = (0.9, 1.1)$ and $(x, y) = (0.95, 0.95)$.

(c) Compare your answers to part (b) to the *actual* values of $f(0.9, 1.1)$ and $f(0.95, 0.95)$.

The *linearization* of the function $f(x,y)$ is the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(this is the same right-hand side as above). The *linear approximation* (or *tangent plane approximation*) is then $f(x,y) \approx L(x,y)$ for (x,y) near (a,b).

3 Here is a more complicated example: $f(x,y) = ye^{xy}$. Calculate $f(1,0)$ and use the *linear approximation* or *tangent plane approximation* from the previous problem to approximate $f(0.9, 0.1)$ and $f(1.1, -0.05)$. Compare your answers to the actual values (if you have a calculator).

4 Sometimes the linear approximation isn't actually very good. Here's an example. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Compute $f_x(0,0)$ and $f_y(0,0)$. You'll need to use the definitions

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \qquad \text{and} \qquad f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

(b) Use the values from part (a) to compute the linear approximation to the surface at the point $(x,y,z) = (0,0,0)$.

(c) Use this approximation to approximate the values of $f(x,y)$ at the points $(x,y) = (a,a)$.

(d) Use this approximation to approximate the values of $f(x,y)$ at the points $(x,y) = (b,-b)$.

(e) Compare the approximations from parts (c) and (d) to the actual values of $f(a,a)$ and $f(b,-b)$. (And notice that these points can be taken to be *very* close to the origin.)

5 We avoid the problem of the last problem by assuming that it doesn't happen. We say that f is *differentiable* if (more or less) the approximation tends to the exact value. There is a theorem that says that f is differentiable at (x_0, y_0) if both f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0). Thus, if the derivatives are nice enough, we always get a good approximation.

Show that this theorem doesn't apply to the previous example by finding general formulas for f_x and f_y and determining that they are not continuous at $(x,y) = (0,0)$.

6 We also want to be able to find tangent planes for parametric surfaces $\mathbf{r}(u, v)$. This problem will step through this for the example

$$\mathbf{r}(u, v) = \langle \cos u, 3\sin u \cos v, 4\sin u \sin v\rangle$$

at the point $(x, y, z) = (0, 0, 4)$ corresponding to $(u, v) = (\frac{\pi}{2}, \frac{\pi}{2})$.

(a) The u grid curve ($v = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(u, \tfrac{\pi}{2}) = \langle \cos u, 3\sin u \cos \tfrac{\pi}{2}, 4\sin u \sin \tfrac{\pi}{2}\rangle = \langle \cos u, 0, 4\sin u\rangle.$$

Find the tangent vector to this curve at $u = \frac{\pi}{2}$.

(b) Repeat part (a) for the v grid curve. That is, the v grid curve ($u = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(\tfrac{\pi}{2}, v) = \langle \cos \tfrac{\pi}{2}, 3\sin \tfrac{\pi}{2} \cos v, 4\sin \tfrac{\pi}{2} \sin v\rangle = \langle 0, 3\cos v, 4\sin v\rangle.$$

Find the tangent vector to this curve at $v = \frac{\pi}{2}$.

(c) Use your answers to parts (a) and (b) to find the normal to the tangent plane at $(x, y, z) = (0, 0, 4)$.

(d) Now find the tangent plane at at $(x, y, z) = (0, 0, 4)$.

(e) The linear approximation is: for (u, v) near (u_0, v_0),

$$\mathbf{r}(u, v) \approx \mathbf{r}(u_0, v_0) + \mathbf{r}_u(u_0, v_0)(u - u_0) + \mathbf{r}_v(u_0, v_0)(v - v_0).$$

Verify that this is the tangent plane approximation you found in part (d).

7 Repeat the previous problem at the point $(u, v) = (\frac{\pi}{2}, 0)$. What point (x, y, z) does this represent?

Approximately Tangent Planes – Answers and Solutions

1 (a) $\mathbf{r}_1'(x_0) = \langle 1, 0, f_x(x_0, y_0)\rangle$

(b) $\mathbf{r}_2'(y_0) = \langle 0, 1, f_y(x_0, y_0)\rangle$

(c) $\langle 1, 0, f_x(x_0, y_0)\rangle \times \langle 0, 1, f_y(x_0, y_0)\rangle = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$

(d) $\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$
or $z = z_0 + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

2 (a) $z = 3 + 2(x - 1) + 4(y - 1)$

(b) $f(0.9, 1.1) \approx 3.2$ $\quad$ $f(0.95, 0.95) \approx 2.7$

(c) $f(0.9, 1.1) = 3.23$ $\quad$ $f(0.95, 0.95) = 2.7075$

3 (Using (a), (b), and (c) in parallel with question 2.)

(a) $z = y$

(b) $f(0.9, 0.1) \approx 0.1$ $\quad$ $f(1.1, -0.05) \approx -0.05$

(c) $f(0.9, 0.1) = 0.1e^{0.09} \approx 0.10941742837052$
$f(1.1, -0.05) = -0.05e^{-0.055} \approx -0.047324257397674$

4 (a) $f_x(0, 0) = f_y(0, 0) = 0$

(b) Thus $L(x, y) = 0$, so the approximation is $z \approx 0$

(c) On this line ($y = x$), we get $f(x, y) = 1/2$ (provided $(x, y) \neq (0, 0)$)

(d) On this line ($y = -x$), we get $f(x, y) = -1/2$ (provided $(x, y) \neq (0, 0)$)

(e) The point is that this is a *bad* approximation, since the approximation error doesn't decrease to zero as $(x, y) \to (0, 0)$.

5 We can differentiate the function in Problem 4 away from the origin pretty easily: it's simply

$$f_x(x, y) = \frac{y\,(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2 y}{(x^2 + y^2)^2}.$$

Thus

$$f_x(x, y) = \begin{cases} \frac{y^3 - x^2 y}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(since we've already calculated the $f_x(0, 0)$ derivative). Similarly

$$f_y(x, y) = \begin{cases} \frac{x^3 - x y^2}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Neither of these is continuous at the origin. For example, along the path $y = 0$, the function $f_y(x, 0) = 1/x$ (for $x \neq 0$) which has no finite limit as $x \to 0$.

6 (a) The u grid curve ($v = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(u, \tfrac{\pi}{2}) = \langle \cos u, 0, 4\sin u \rangle,$$

so the tangent vector is the derivative of this with respect to u:

$$\frac{\partial \mathbf{r}}{\partial u}(u, \tfrac{\pi}{2}) = \mathbf{r}_u(u, \tfrac{\pi}{2}) = \langle -\sin u, 0, 4\cos u \rangle.$$

When $u = \frac{\pi}{2}$, this is the vector $\mathbf{r}_u(\frac{\pi}{2}, \frac{\pi}{2}) = \langle -\sin\frac{\pi}{2}, 0, 4\cos\frac{\pi}{2} \rangle = \langle -1, 0, 0 \rangle$.

(b) Since $\mathbf{r}_v(\frac{\pi}{2}, v) = \langle 0, -3\sin v, 4\cos v \rangle$, the tangent vector to the v grid curve at $v = \frac{\pi}{2}$ is $\mathbf{r}_v(\frac{\pi}{2}, \frac{\pi}{2}) = \langle 0, -3\sin\frac{\pi}{2}, 4\cos\frac{\pi}{2} \rangle = \langle 0, -3, 0 \rangle$.

(c) The normal to the tangent plane at $(x, y, z) = (0, 0, 4)$ is

$$\mathbf{r}_u(\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times \mathbf{r}_v(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = \langle -1, 0, 0 \rangle \times \langle 0, -3, 0 \rangle = \langle 0, 0, 3 \rangle.$$

(d) The tangent plane at at $(x, y, z) = (0, 0, 4)$ is $\langle 0, 0, 3 \rangle \cdot \langle x, y, z - 4 \rangle = 0$ or $3(z - 4) = 0$ or $z = 4$.

(e) The linear approximation is: for (u, v) near $(\frac{\pi}{2}, \frac{\pi}{2})$,

$$\mathbf{r}(u, v) \approx \mathbf{r}(\tfrac{\pi}{2}, \tfrac{\pi}{2}) + \mathbf{r}_u(\tfrac{\pi}{2}, \tfrac{\pi}{2})(u - \tfrac{\pi}{2}) + \mathbf{r}_v(\tfrac{\pi}{2}, \tfrac{\pi}{2})(v - \tfrac{\pi}{2}).$$

or

$$\mathbf{r}(u, v) \approx \langle 0, 0, 4 \rangle + \left(u - \tfrac{\pi}{2}\right) \langle -1, 0, 0 \rangle + \left(v - \tfrac{\pi}{2}\right) \langle 0, -3, 0 \rangle.$$

This is the tangent plane approximation you found in part (d): this says $x = -(u - \frac{\pi}{2})$, $y = -3(v - \frac{\pi}{2})$, and $z = 4$. Thus x and y are arbitrary and $z = 4$.

7 The point $(u, v) = (\frac{\pi}{2}, 0)$ represents the point $(x, y, z) = (0, 3, 0)$.

(a) The u grid curve ($v = 0$) is given by the parametric curve

$$\mathbf{r}(u, 0) = \langle \cos u, 3\sin u, 0 \rangle,$$

so the tangent vector is the derivative of this with respect to u:

$$\frac{\partial \mathbf{r}}{\partial u}(u, 0) = \mathbf{r}_u(u, 0) = \langle -\sin u, 3\cos u, 0 \rangle.$$

When $u = \frac{\pi}{2}$, this is the vector $\mathbf{r}_u(\frac{\pi}{2}, 0) = \langle -\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2}, 0 \rangle = \langle -1, 0, 0 \rangle$.

(b) Since $\mathbf{r}(\frac{\pi}{2}, v) = \langle 0, 3\cos v, 4\sin v \rangle$, we get $\mathbf{r}_v(\frac{\pi}{2}, v) = \langle 0, -3\sin v, 4\cos v \rangle$ and $\mathbf{r}_v(\frac{\pi}{2}, 0) = \langle 0, 0, 4 \rangle$.

(c) The normal to the tangent plane at $(x, y, z) = (0, 3, 0)$ is

$$\mathbf{r}_u(\tfrac{\pi}{2}, 0) \times \mathbf{r}_v(\tfrac{\pi}{2}, 0) = \langle -1, 0, 0\rangle \times \langle 0, 0, 4\rangle = \langle 0, 4, 0\rangle.$$

(d) The tangent plane at at $(x, y, z) = (0, 3, 0)$ is $\langle 0, 4, 0\rangle \cdot \langle x, y-3, z\rangle = 0$ or $4(y-3) = 0$ or $y = 3$.

(e) The linear approximation is: for (u, v) near $(\frac{\pi}{2}, 0)$,

$$\mathbf{r}(u, v) \approx \mathbf{r}(\tfrac{\pi}{2}, 0) + \mathbf{r}_u(\tfrac{\pi}{2}, 0)(u - \tfrac{\pi}{2}) + \mathbf{r}_v(\tfrac{\pi}{2}, 0)(v - 0)$$

or

$$\mathbf{r}(u, v) \approx \langle 0, 3, 0\rangle + (u - \tfrac{\pi}{2})\langle -1, 0, 0\rangle + v\langle 0, 0, 4\rangle.$$

This is the tangent plane approximation you found in part (d): this says $x = -(u - \frac{\pi}{2})$, $y = 3$, and $z = 4v$. Thus x and z are arbitrary and $y = 3$.