Directional Derivatives

1 If $f(x, y) = 3x + 7y^2$ and $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$, find the directional derivative $D_{\mathbf{u}}f$ at $(1, 1)$.

2 If $f(x, y) = \sin(xy)$, find the directional derivative $D_{\mathbf{u}}f$ where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

3 You are skiing on a mountain which happens to be the graph of the function $f(x, y) = 10 - x^2 - y^4$. You are at the point $(1, 1, 8)$. If you want to ski down the steepest path, what direction should you head?

4 A fly is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly?

5 Alice the "A" student is debating Chuck the "C" student. Alice says that the direction of greatest ascent on the graph $z = f(x, y)$ is in the direction ∇f. Chuck says that instead we should look at the level surface $F(x, y, z) = z - f(x, y) = 0$ and go in direction $\operatorname{grad} F$. Who is right? How would you explain this to the student that is wrong?

6 A racehorse lives in a valley which happens to be the graph of $f(x, y) = 3x^2 + y^2$. He is doomed to wander his racetrack, which is the set of points in the valley where $x^2 + y^2 = 1$.

The racehorse secretly wishes to be a mountain climber, and his fantasy is to escape from his racetrack and take the steepest path up the mountain. At what point should he make his escape, and in what direction should he run? (There are actually two possible answers.)

Directional Derivatives – Solutions

1 We use the formula $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Here $\nabla f = \langle 3, 14y \rangle$, so $\nabla f(1,1) \cdot \mathbf{u} = \langle 3, 14 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = 13$.

2 Now $\nabla f = \langle y\cos(xy), x\cos(xy) \rangle$, so $\nabla f \cdot \mathbf{u} = \frac{x+y}{\sqrt{2}} \cos(xy)$.

3 Since the problem asks only for a direction, we should give our answer as a unit vector $\mathbf{u}$. We are looking for the unit vector $\mathbf{u}$ for which $D_{\mathbf{u}}f(1,1)$ is the most negative. We know that $D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \cdot \mathbf{u}$. Let's first calculate ∇f: it is $\langle -2x, -4y^3 \rangle$, so $\nabla f(1,1) = \langle -2, -4 \rangle$. Therefore, we want to find the unit vector for which $\langle -2, -4 \rangle \cdot \mathbf{u}$ is the most negative.

Remember that the dot product of two vectors is given by the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$.

If θ is the angle between $\langle -2, -4 \rangle$ and $\mathbf{u}$, then $|\langle -2, -4 \rangle \cdot \mathbf{u}| = |\langle -2, -4 \rangle|\,|\mathbf{u}| \cos\theta$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$, so we really want to make $\cos\theta$ as negative as possible. This means we should take $\theta = \pi$, so we want to pick the unit vector which goes in the direction opposite of $\langle -2, -4 \rangle$. That is, we want a vector which goes in the direction of $\langle 2, 4 \rangle$ but which has length 1. To get such a vector, we just divide $\langle 2, 4 \rangle$ by its length, which is $2\sqrt{5}$. So, our answer is $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

4 We want to find the unit vector $\mathbf{u}$ which maximizes $D_{\mathbf{u}}T(1,1,1)$. Using the same idea as in the last problem, this should be the unit vector in the direction of $\nabla T(1,1,1)$.

Since $\nabla T = \langle 2x, 4y^3, 4z \rangle$, we want the unit vector in the direction of $\langle 2, 4, 4 \rangle$, which is $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$.

5 Alice is correct: the direction of greatest ascent on the surface $z = f(x,y)$ is in the direction ∇f.

What is the problem with Chuck's answer? He's looking at the level surface $F(x,y,z) = 0$, where $F(x,y,z) = z - f(x,y)$. This is the same surface that Alice is looking at. But when he suggests moving in the ∇F direction, this means moving a direction in space that increases the value of $F(x,y,z)$ as quickly as possible. In particular, this means moving off the level surface $F = 0$. This is a different problem altogether.

6 What the problem is really asking us to do is find the points (x,y) and unit vectors $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(x,y)$ is biggest, although we're only allowed to look at points where $x^2 + y^2 = 1$.

Let's first focus on a specific point (x,y) and figure out the biggest the directional derivative $D_{\mathbf{u}}f(x,y)$ could be at that point. We know that the directional derivative is largest in the direction of the gradient. The unit vector in the direction of the gradient is $\frac{\nabla f}{|\nabla f|}$, and the directional derivative in this direction is $\nabla f \cdot \frac{\nabla f}{|\nabla f|}$. Since $\nabla f \cdot \nabla f$ is just $|\nabla f|^2$, the directional derivative in this direction is really $|\nabla f|$. Thus, we could restate the problem as: find the points (x,y) for which $|\nabla f(x,y)|$ is largest.

Let's now calculate the gradient: $\nabla f(x,y) = \langle 6x, 2y \rangle$, so $|\nabla f(x,y)| = \sqrt{36x^2 + 4y^2}$.

Thus, we can again restate the problem as: maximize $36x^2 + 4y^2$ subject to the constraint that $x^2 + y^2 = 1$. Next week, we will learn a method for doing this called the method of Lagrange multipliers. However, we can figure this problem out without Lagrange multipliers.

We can write $36x^2 + 4y^2$ as $32x^2 + 4(x^2 + y^2)$; when $x^2 + y^2 = 1$, this is just equal to $32x^2 + 4$. So, we're really trying to maximize $32x^2 + 4$. We do this by making x^2 as large as possible.

Since (x, y) has to stay on the circle $x^2 + y^2 = 1$, this means we want $x = \pm 1$, which makes $y = 0$.

The unit vectors $\mathbf{u}$ are supposed to go in the direction of $\nabla f = \langle 6x, 2y \rangle$. In the case of the point $(1, 0)$, this means $\mathbf{u} = \langle 1, 0 \rangle$; in the case of the point $(-1, 0)$, this means $\mathbf{u} = \langle -1, 0 \rangle$.

So our answer is: the point $(1, 0)$ with the unit vector $\langle 1, 0 \rangle$ and the point $(-1, 0)$ with the unit vector $\langle -1, 0 \rangle$.