Extremal Points

Tests for Maxima, Minima, Saddle Points

Critical Points:

A point (a, b) is a *critical point* for the function $z = f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

That is, critical points are those points where $\nabla f = \langle f_x, f_y \rangle = \mathbf{0}$.

Is A Critical Point A Max, Min, or Saddle?
Set $D = f_{xx} f_{yy} - f_{xy}^2$.

- $D < 0$ at (a, b) $\implies$ (a, b) is a saddle point.
- $D > 0$ at (a, b)
  - $f_{xx} < 0$ at (a, b) $\implies$ (a, b) a local max
  - $f_{xx} > 0$ at (a, b) $\implies$ (a, b) a local min
- $D = 0$ at (a, b) $\implies$ no info about (a, b)

For each of the following functions...

(a) Compute f_x, f_y, f_{xx}, f_{yy}, and f_{xy}. (You can assume that $f_{yx} = f_{xy}$.)

(b) Find the critical points. That is, find all points where both $f_x = 0$ and $f_y = 0$.

(c) Find the value of f_{xx} and D at each critical point.

(d) Using the test above, determine (if possible) whether each critical point is a local maximum, a local minimum, or a saddle point.

1 $f(x, y) = x^2 + 2xy + 2y^2 - 8y + 12$

2 $f(x, y) = x^2 - 2y^2 + xy - 4$

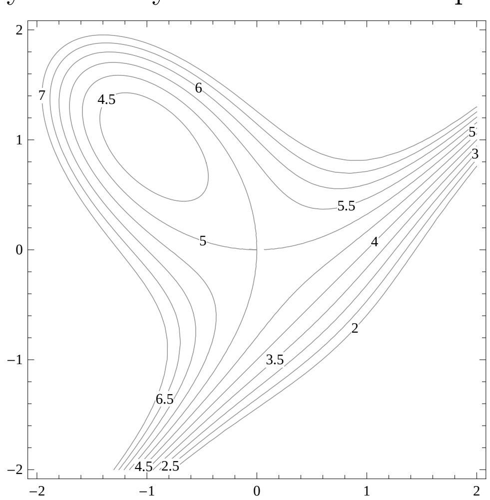
3 $f(x, y) = 8 - x^2 - xy - y^2$

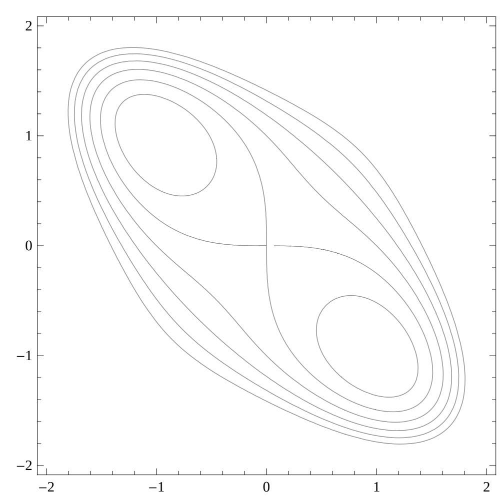
4 $f(x, y) = x^4 + y^4 + 4xy + 3$

$\boxed{5}$ $f(x,y) = y^2 + 2xy - x^2 + 2x - 2y + 3$

$\boxed{6}$ $f(x,y) = 5 - x^3 + y^3 + 3xy$

$\boxed{7}$ Two of the above functions are shown as contour plots below. For each plot, predict the location of the critical points and classify them as maxima, minima, or saddle points. Can you identify which functions are plotted here?





Answers:

$\boxed{1}$ (b) $(a,b) = (-4,4)$ (c) $f_{xx} = 2$, $D = 4$ (d) min

$\boxed{2}$ (b) $(a,b) = (0,0)$ (c) $f_{xx} = 2$, $D = -9$ (d) saddle

$\boxed{3}$ (b) $(a,b) = (0,0)$ (c) $f_{xx} = -2$, $D = 3$ (d) max

$\boxed{4}$ (b) $(a,b) = (0,0)$, $(1,-1)$, $(-1,1)$

(d) minima at $\pm(1,-1)$, saddle at origin

$\boxed{5}$ (b) $(a,b) = (1,0)$ (c) $f_{xx} = -2$, $D = -8$ (d) saddle

$\boxed{6}$ (b) $(a,b) = (0,0)$ & $(-1,1)$ (d) saddle & min

$\boxed{7}$ These plots are Problems 6 (left) and 4 (right)