Global Extrema; Lagrange Multipliers II

Here's the theorem we're going to take advantage of today:

Extreme Value Theorem
(For Functions Of Two Variables)

If $f(x, y)$ is continuous on a closed, bounded region D in the plane, then f attains a maximum value $f(x_1, y_1)$ and a minimum value $f(x_2, y_2)$ at points (x_1, y_1) and (x_2, y_2) in D.

$\boxed{1}$ For which of the regions D described below is it true that every continuous function $f(x, y)$ must attain an absolute maximum value and absolute minimum value on D? (There may be more than one.)

(i) D is the set of points (x, y) such that $|x| \leq 4$ and $|y| < 2$.

(ii) D is the set of points (x, y) such that $|x + y| \leq 1$.

(iii) D is the set of points (x, y) such that $x^2 + 4y^2 \leq 1$.

(iv) D is the set of points (x, y) such that $x^2 + 4y \leq 1$.

(v) D is the set of points (x, y) such that $-x \leq y \leq x$.

Now do the following:

(a) For one region D that you picked, find the absolute minimum and absolute maximum value of $f(x, y) = x^2 - 4x + y^2$ on the region.

(b) For one region you didn't pick, find a function $f(x, y)$ which has either no maximum or no minimum on the region D.

$\boxed{2}$ Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2$ in the elliptical region $2x^2 + 4y^2 \leq 1$.

$\boxed{3}$ Find the absolute maximum and minimum of the function $f(x, y) = x^2 - y^2$ in the circular region $x^2 + y^2 \leq 1$.

$\boxed{4}$ Find the point (or points) on the graph of the function $f(x, y) = 2x^2 - y^2 + 1$ that is closest to the origin. (You may make the reasonable assumption that $x^2 + y^2 \leq 10$.)

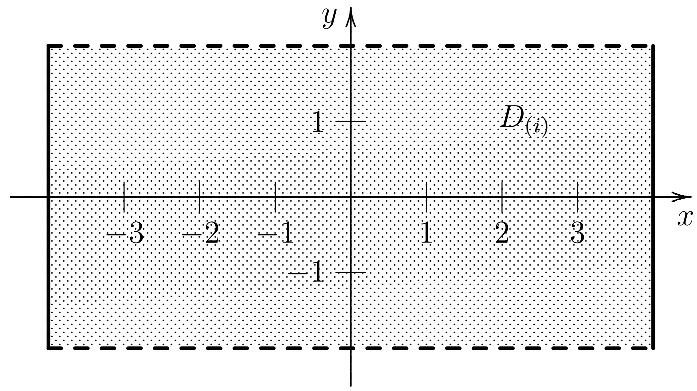
$\boxed{5}$ Find the absolute maximum and minimum of the function $f(x, y) = 3xy - y + 5$ in the circular region $x^2 + y^2 \leq 1$.

$\boxed{6}$ The temperature in a room is described by the function $T(x, y, z) = x^2y + z$. A bug is walking on a surface in the room, which can be described parametrically by $\mathbf{r}(u, v) = \langle u, e^v, u + v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq 1$. What is the warmest point the bug can reach? What is the coolest?

Solutions – Global Extrema

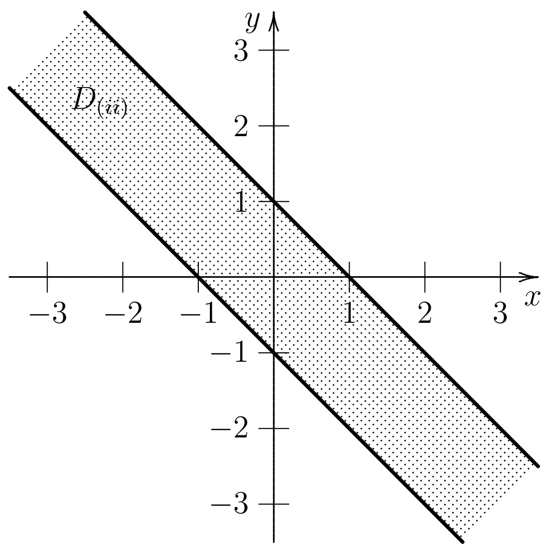
$\boxed{1}$ We begin by determining whether or not our theorem applies to each region. That is, we check to see whether each region is closed and bounded.

(i) Here D is the set of points in a rectangle:



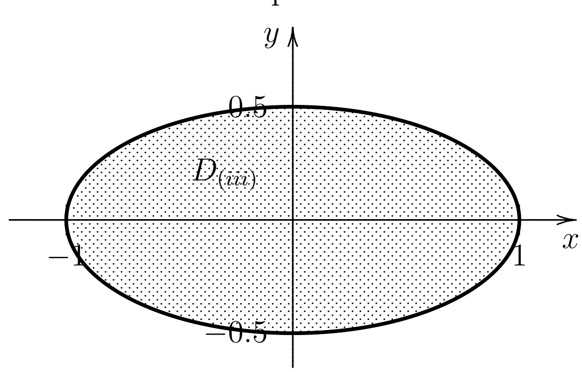
This is clearly bounded, but it is not closed as the boundary points along the line segments $y = 2$ and $y = -2$ are not part of D.

(ii) Here D is the set of points between the lines $x + y = 1$ and $x + y = -1$:



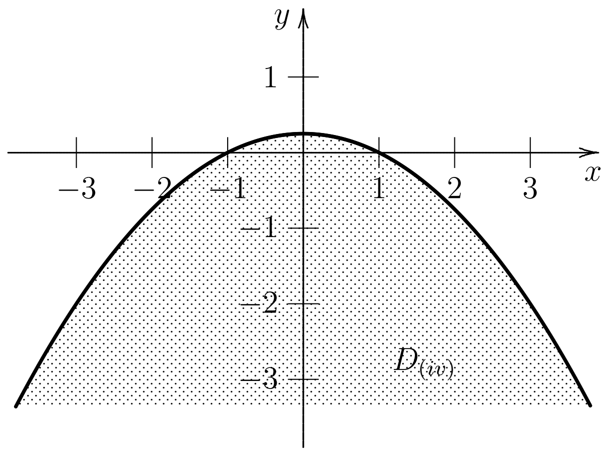
This is closed but not bounded.

(iii) Here D is the set of points on or in an ellipse:



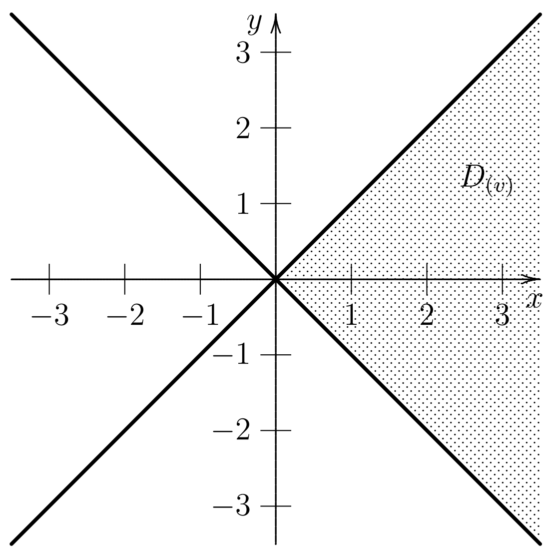
This is both closed and bounded.

(iv) Here D is the set of points on or below the parabola $y = \frac{1-x^2}{4}$:



This is closed but not bounded.

(v) Here D is the set of points above the line $y = -x$ and also below the line $y = x$:



This is closed but not bounded.

(a) Our only option is (iii). We'll look at a collection of points:

- Points inside the ellipse that might be local maxima or local minima. These critical points are where $\nabla f = \mathbf{0}$.
- Points on the ellipse that might be local maxima or local minima of $f(x, y)$ when constrained to the ellipse. To find these we use Lagrange multipliers with $g(x, y) = x^2 + 4y^2 = 1$.

Since $\nabla f = \langle 2x - 4, 2y\rangle$, the only point where $\nabla f = \mathbf{0}$ is the point $(x, y) = (2, 0)$. This point is outside the ellipse and so can be ignored.

We turn to finding points on the ellipse using Lagrange multipliers. From $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, we have three equations:

$$2x - 4 = \lambda 2x \qquad 2y = \lambda 8y \qquad x^2 + 4y^2 = 1.$$

From the first equation we see that $x = 0$ or $\lambda = 1 - \frac{2}{x}$. The second equation implies $y = 0$ or $\lambda = 1/4$. So if $\lambda \neq 1/4$, then $y = 0$ and so (from the last equation) $x = \pm 1$. If $\lambda = 1/4$, then $x = \frac{8}{3}$ and so the last equation cannot be satisfied (as $x^2 > 1$ already). Thus we have only two points: $(x, y) = (\pm 1, 0)$. Since $f(1, 0) = -3$ and $f(-1, 0) = 5$, our absolute maximum is at $(x, y) = (-1, 0)$ and our absolute minimum is at $(1, 0)$.

(b) We'll find examples of functions without maxima or minima for each of the remaining regions.

(i) This one is very simple: the function $f(x, y) = y$ has no maximum or minimum in $D_{(i)}$. This function attains every value between -2 and $+2$, but neither of these endpoints.

(ii) Consider the function $f(x, y) = y - x$. The level sets for this f are lines parallel to the line $y = x$, each of which is perpendicular to the defining lines of the region $D_{(ii)}$. As we move up and to the left on $D_{(ii)}$, the values of f increase without bound; as we move down and to the right, f decreases without bound. (The level curve $f(x, y) = k$ is the line $y = x + k$, so the value k is simply the y-intercept of this line. We can make this as large or small as we like.) Thus f has no maximum or minimum on $D_{(ii)}$.

(iv) Here the function $f(x, y) = x$ has no maximum or minimum.

(v) Here the function $f(x, y) = y$ has no maximum or minimum. This example and the previous one are simple examples of why the theorem requires that D be bounded.

$\boxed{2}$ There is one critical point – the origin – in the interior where $\nabla f = \langle 2x, 2y \rangle = \mathbf{0}$. When constrained to the boundary $2x^2 + 4y^2 = 1$, there are four points where $\nabla f = \lambda \nabla g$, or $\langle 2x, 2y \rangle = \lambda \langle 4x, 8y \rangle$. These points are $(x, y) = (\pm 1/\sqrt{2}, 0)$ and $(x, y) = (0, \pm 1/2)$. Our maximum is $f\left(\pm 1/\sqrt{2}, 0\right) = 1/2$ and our minimum is $f(0, 0) = 0$.

Thomas K. (the apprentice in the 11:00 class) pointed out that this is equivalent to minimizing and maximizing the distance from the origin to points in the elliptical region.

$\boxed{3}$ In this problem the one critical point – again the origin – is a saddle point. The absolute maximum and minimum lie on the boundary where $\nabla f = \lambda \nabla g$ or $\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$. This can occur only when $x = 0$ or $y = 0$, which gives us the points $(x, y) = (0, \pm 1)$ and $(x, y) = (\pm 1, 0)$. The maximum is $f(\pm 1, 0) = 1$ and the minimum is $f(0, \pm 1) = -1$.

$\boxed{4}$ Here we're looking for the points $(x, y, z) = (x, y, 2x^2 - y^2 + 1)$ that minimize $\sqrt{x^2 + y^2 + z^2}$ or, even better, minimize $x^2 + y^2 + z^2$. Thus we'd like to minimize $F(x, y) = x^2 + y^2 + (2x^2 - y^2 + 1)^2$. This can be done in the usual way.

Of course, it's probably easier to say: we'd like to minimize $F(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 2x^2 - y^2 + 1 - z = 0$. This is a problem built for Lagrange multipliers: $\nabla F = \lambda \nabla g$ when

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 4x, -2y, -1 \rangle.$$

These are the equations (with the constraint)

$$\begin{aligned} 2x &= 4x\lambda \\ 2y &= -2y\lambda \\ 2z &= -\lambda \\ 2x^2 - y^2 + 1 - z &= 0. \end{aligned}$$

The first equation gives us two cases: $x = 0$ or $\lambda = 1/2$. We'll consider these in turn.

Case 1: $x = 0$ In this case, the second equation now says that either $y = 0$ or $\lambda = -1$. If $y = 0$, the last equation tells us that $z = 1$, so we get the point $(x, y, z) = (0, 0, 1)$. If $\lambda = -1$, the third equation tells us that $z = 1/2$, so the last equation now implies $y = \pm 1/\sqrt{2}$. Thus we get the points $(x, y, z) = \left(0, \pm 1/\sqrt{2}, 1/2\right)$.

Case 2: $\lambda = 1/2$ In this case, the second and third equations allow us to solve for $y = 0$ and $z = -1/4$. The final equation now says that $x^2 = -5/8$, so we get no critical points at all.

We now compute the value of $F = x^2 + y^2 + z^2$ at these three points to find the minimum:

$$F(0, 0, -1) = 1 \qquad \text{and} \qquad F(0, \pm 1/\sqrt{2}, 1/2) = 3/4.$$

The smallest of these is the pair of points $(0, \pm 1/\sqrt{2}, 1/2)$.

$\boxed{5}$ I end up with four points:

$$\left(\frac{1+\sqrt{73}}{12}, \pm\sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx (0.795, \pm 0.606)$$

and

$$\left(\frac{1-\sqrt{73}}{12}, \pm\sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx (-0.629, \pm 0.778)\,.$$

Note that

$$f\left(\frac{1+\sqrt{73}}{12}, \sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx 5.840 \qquad \text{and} \qquad f\left(\frac{1+\sqrt{73}}{12}, -\sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx 4.160,$$

while

$$f\left(\frac{1-\sqrt{73}}{12}, \sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx 2.756 \qquad \text{and} \qquad f\left(\frac{1-\sqrt{73}}{12}, -\sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx 7.244.$$

This means the maximum is almost 7.25 and the minimum is just over 2.75.

$\boxed{6}$ One simple way to do this is to maximize the function $f(u,v) = T(\mathbf{r}(u,v))$ on the unit square $0 \le u \le 1$, $0 \le v \le 1$. This turns out to be the function $f(u,v) = u^2e^v + u + v$. Notice that $\nabla f = \langle 2ue^v + 1, u^2e^v + 1\rangle$. Since u and v cannot be negative, neither of these components can be zero. Thus there are no critical points inside this square.

On the boundary, we could try to use Lagrange multipliers, but this won't work. On our boundaries, ∇g will be parallel to $\mathbf{i}$ or $\mathbf{j}$, and ∇f cannot be parallel to these vectors. (This is for the same reason as above: the components of ∇f cannot be zero on our square.) What goes wrong? Well, our boundary has corners so the tangent line doesn't exist at these points. So while these corners might be maxima or minima, we won't have ∇f and ∇g parallel there.

It turns out that if we look carefully at the boundaries, we see that the absolute maximum occurs at $(u,v) = (1,1)$ and the absolute minimum occurs at $(u,v) = (0,0)$. These correspond to the points $\mathbf{r}(0,0) = \langle 0,1,0\rangle$ and $\mathbf{r}(1,1) = \langle 1,e,2\rangle$. The temperatures at these points are $T(0,1,0) = 0$ (the coolest point) and $T(1,e,2) = e+2$ (the warmest point).

How do we "look carefully at the boundaries"? If we restrict ourselves to each piece of the boundary in turn, we have four functions:

The $v = 0$ piece:	$f(u,0) = u^2 + u$	min: 0, max 2
The $v = 1$ piece:	$f(u,1) = eu^2 + u + 1$	min: 1, max $e + 2$
The $u = 0$ piece:	$f(0,v) = v$	min: 0, max 1
The $u = 1$ piece:	$f(1,v) = e^v + 1 + v$	min: 2, max $e + 2$

We can thus see that the maximum occurs when $(u,v) = (1,1)$ and the minimum occurs when $(u,v) = (0,0)$.