Double Integrals

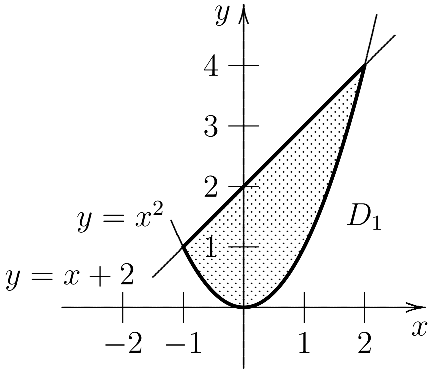
A region D is called Type I if it can be written in the following way:

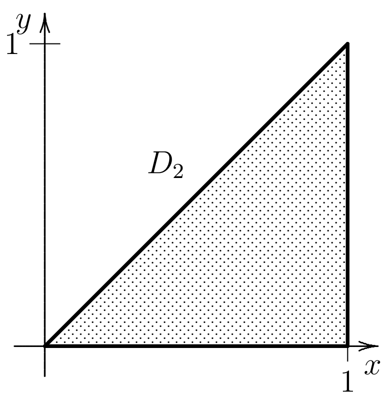
$$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}.$$

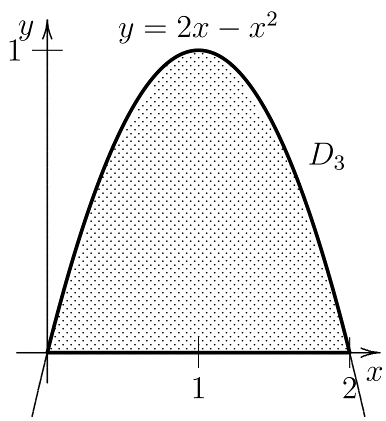
We can then compute a double integral as

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Here are some Type I regions. Compute the integrals in the problems below.







$\boxed{1}$ $\displaystyle\iint_{D_1} xy\, dA$

$\boxed{2}$ $\displaystyle\iint_{D_2} e^{x^2}\, dA$

$\boxed{3}$ $\displaystyle\iint_{D_3} (x-1)y^2\, dA$

$\boxed{4}$ $\displaystyle\iint_{D_1} 1\, dA$

Similarly, a region D is called Type II if it can be written in the following way:

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}.$$

We can then compute a double integral as

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

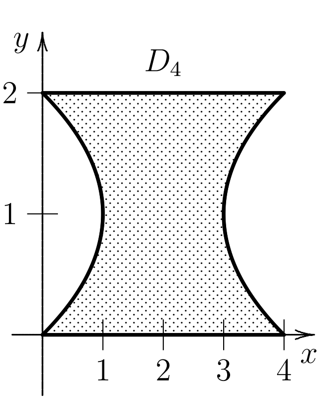
Compute the following integrals as Type II integrals. Some of the regions are shown on the next page.

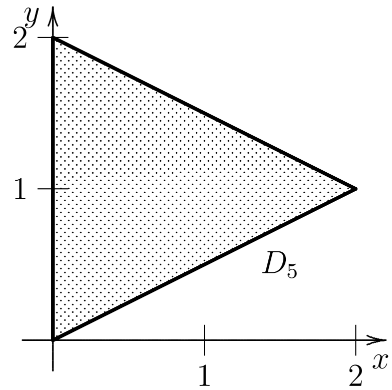
$\boxed{5}$ $\displaystyle\iint_{D_2} (1-y)^3\, dA$

$\boxed{6}$ $\displaystyle\iint_{D_4} (y-1)x^2\, dA$

$\boxed{7}$ $\displaystyle\iint_{D_2} \cos(x^2)\, dA$

$\boxed{8}$ $\displaystyle\iint_{D_5} (x-1)\, dA$





(The curves in D_4 are $x = 2y - y^2$ and $x = y^2 - 2y + 4$.)

Of course, sometimes it is necessary to draw the region and possibly even switch the order of integration! For each of the following integrals, draw the region in question, write down an integral with the reverse order of integration, then finally integrate.

$\boxed{9}$ $\displaystyle\int_0^2 \int_x^2 (x + y)\, dy\, dx$

$\boxed{10}$ $\displaystyle\int_0^2 \int_0^{\sqrt{x}} (x^2 - y^2)\, dy\, dx$

The real trouble begins when you can't integrate without switching the order of integration. Here are four examples of this. You should draw the region of integration!

$\boxed{11}$ $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

$\boxed{12}$ $\displaystyle\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x}\, dx\, dy$

$\boxed{13}$ $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \left(3x - x^3\right)^{10}\, dx\, dy$

$\boxed{14}$ $\displaystyle\int_1^{e^3} \int_{\ln(y)}^3 \left(e^x - x\right)^5\, dx\, dy$

Double Integrals – Answers and Solutions

$\boxed{1}$ $$\iint_{D_1} xy\, dA = \int_{-1}^{2}\int_{x^2}^{x+2} xy\, dy\, dx = \frac{45}{8}$$

$\boxed{2}$ $$\iint_{D_2} e^{x^2}\, dA = \int_{0}^{1}\int_{0}^{x} e^{x^2}\, dy\, dx = \frac{1}{2}(e-1)$$

$\boxed{3}$ $$\iint_{D_3} (x-1)y^2\, dA = \int_{0}^{2}\int_{0}^{2x-x^2} (x-1)y^2\, dy\, dx = 0$$

As pointed out in class, the function $f(x, y)$ is anti-symmetric about the line $x = 1$, so when we integrate over a region that is symmetric about $x = 1$ we should expect to get zero.

$\boxed{4}$ $$\iint_{D_1} 1\, dA = \int_{-1}^{2}\int_{x^2}^{x+2} 1\, dy\, dx = \int_{-1}^{2} \left[(x+2) - x^2\right]\, dy\, dx = 4.5$$

Notice that this is the area of the region D_1. In the middle of this computation we see the single-variable calculus formula for the area between two curves (the integral of "the top curve minus the bottom curve").

$\boxed{5}$ $$\iint_{D_2} (1-y)^3\, dA = \int_{0}^{1}\int_{y}^{1} (1-y)^3\, dx\, dy = \int_{0}^{1} (1-y)^4\, dy = -\frac{1}{5}(1-y)^5\Big|_0^1 = \frac{1}{5}$$

$\boxed{6}$ $$\iint_{D_4} (y-1)x^2\, dA = \int_{0}^{2}\int_{2y-y^2}^{y^2-2y+4} (y-1)x^2\, dx\, dy$$
$$= \frac{1}{3}\int_{0}^{2} (y-1)\left[\left(y^2-2y+4\right)^3 - \left(2y-y^2\right)^3\right]\, dy = 0$$

It's easy to compute this last integral using the substitution $u = y^2 - 2y$. Once we make this substitution we see that the limits of integration are from $u = 0$ to $u = 0$; hence the integral is zero. We could also have used an argument similar to that of Problem 3 to show that this is zero.

$\boxed{7}$ $$\iint_{D_2} \cos(x^2)\, dA = \frac{1}{2}\sin(1)$$

The point of this problem is to illustrate that if we write it as a Type II integral, as asked, we get an integral we just can't compute:

$$\iint_{D_2} \cos(x^2)\, dA = \int_{0}^{1}\int_{y}^{1} \cos(x^2)\, dx\, dy.$$

On the other hand, if we re-write it as a Type I integral, everything works out nicely:

$$\iint_{D_2} \cos(x^2)\, dA = \int_{0}^{1}\int_{0}^{x} \cos(x^2)\, dy\, dx = \int_{0}^{1} x\cos(x^2)\, dx = \frac{1}{2}\sin(x^2)\Big|_0^1 = \frac{1}{2}\sin(1).$$

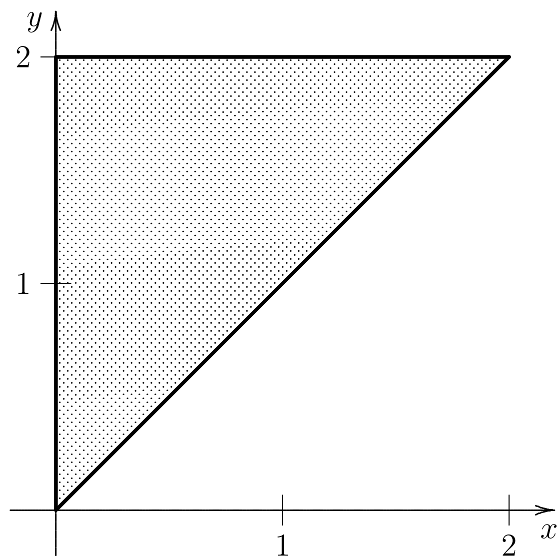
$\boxed{8}$ $$\iint_{D_5} (x-1)\,dA = \int_1^2 \int_0^{4-2y} (x-1)\,dx\,dy + \int_0^1 \int_0^{2y} (x-1)\,dx\,dy = -\frac{1}{3} + \left(-\frac{1}{3}\right) = -\frac{2}{3}$$

Note that it would be easy to write this as only one integral if we thought of D_5 as a Type I region:

$$\iint_{D_5} (x-1)\,dA = \int_0^2 \int_{x/2}^{2-x/2} (x-1)\,dy\,dx = -\frac{2}{3}.$$

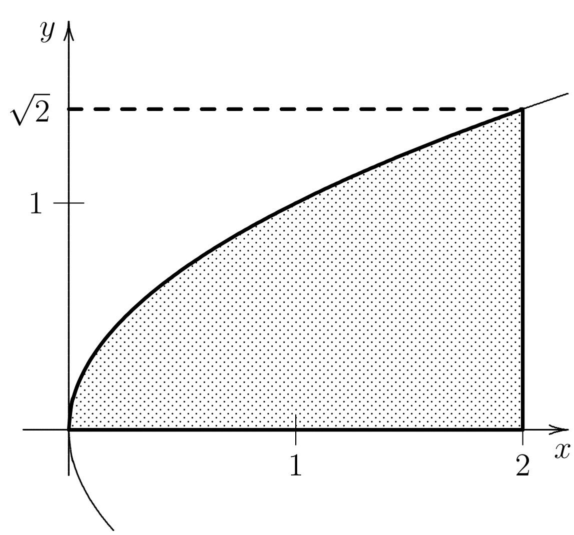
$\boxed{9}$ $$\int_0^2 \int_x^2 (x+y)\,dy\,dx = \int_0^2 \int_0^y (x+y)\,dx\,dy = 4$$

Here's a picture of the region over which we're integrating:

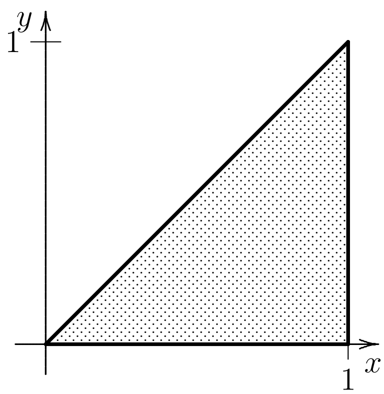


$\boxed{10}$ $$\int_0^2 \int_0^{\sqrt{x}} (x^2 - y^2)\,dy\,dx = \int_0^{\sqrt{2}} \int_{y^2}^2 (x^2 - y^2)\,dx\,dy = \frac{184}{105}\sqrt{2} \approx 2.4782.$$

Here's a picture of the region over which we're integrating:



$\boxed{11}$ The region of integration is

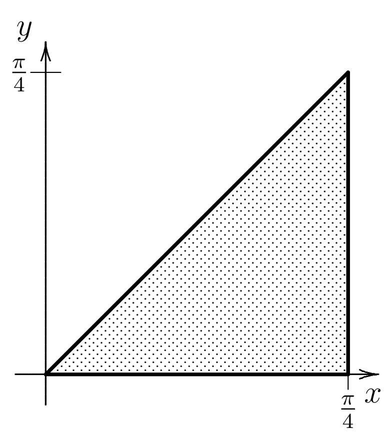


Thus the integral can be re-written as

$$\int_0^1 \int_0^x e^{x^2} \, dy \, dx = \int_0^1 x e^{x^2} \, dx = \frac{1}{2}(e-1) \, .$$

This integral turn out to be the same integral as in Problem 2.

$\boxed{12}$ The region of integration is

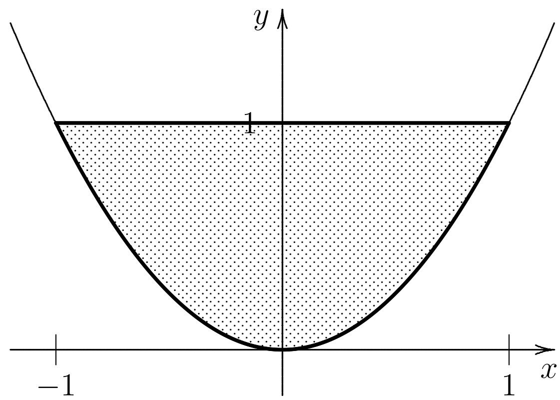


Thus the integral can be re-written as

$$\int_0^{\pi/2} \int_0^x \frac{\sin(x)}{x} \, dy \, dx = \int_0^1 \sin(x) \, dx = 1.$$

Note that the function $\frac{\sin(x)}{x}$ is undefined at $x = 0$, which in this region is simply the origin. But since $\frac{\sin(x)}{x} \to 1$ as $x \to 0$, we can extend this integrand to be continuous at this point. Let's assume that this is what we've done.

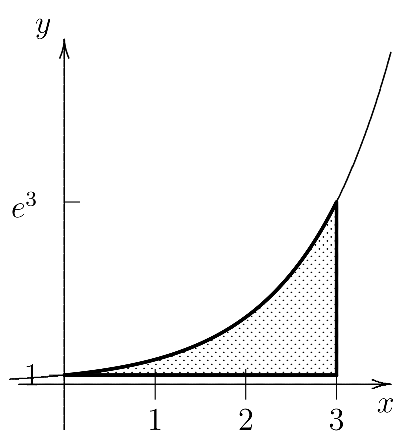
$\boxed{13}$ The region of integration is



Thus we can write the integral as

$$\int_{-1}^{1}\int_{x^2}^{1}\left(3x-x^3\right)^{10}\ dy\ dx=\int_{-1}^{1}\left(3x-x^3\right)^{10}(1-x^2)\ dx=\frac{2^{12}}{33}.$$

$\boxed{14}$ The region of integration is



Thus we can write the integral as

$$\int_{0}^{3}\int_{1}^{e^x}(e^x-x)^5\ dy\ dx=\int_{0}^{3}(e^x-x)^5(e^x-1)\ dx=\frac{1}{6}\left(e^3-4\right).$$