Double Integrals in Polar Coordinates

For Problems 1–6,

(a) Sketch the region of integration.

(b) Try to describe the region in polar coordinates and decide whether you should use polar coordinates or rectangular coordinates.

(c) Evaluate the integral. If you are using polar coordinates, remember that $dA = r\,dr\,d\theta$.

1 $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region $x^2+y^2 \le 1$.

2 $\iint_R x\,dA$, where R is the region $x^2+y^2 \le 1$, $x \ge 0$.

3 $\iint_R (x+y)^2\,dA$, where R is the region $1 \le x^2+y^2 \le 9$, $x \ge 0$, $y \ge 0$.
Hint: Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

4 $\iint_R \left(x^2+y^2\right)\,dA$, where R is the region $0 \le x \le 1$, $0 \le y \le 1$.

5 $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy\, dx$

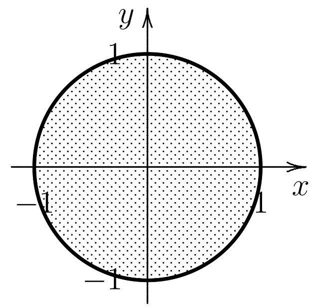
6 $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\, dx\, dy$

7 Compute the volume of the solid that is under the paraboloid $z = 9 - x^2 - y^2$ and above the xy-plane.

8 Compute the volume of the part of the sphere $x^2 + y^2 + (z+1)^2 = 4$ which is above the xy-plane.

Double Integrals in Polar Coordinates – Solutions

1 (a) The region of integration is the unit circle:

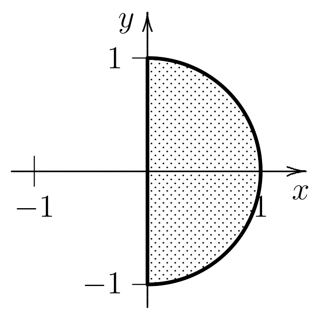


(b) In polar coordinates, the region is $0 \le \theta \le 2\pi$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{2\pi} \int_0^1 r\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^1 r^2\ dr\ d\theta = \int_0^{2\pi} \frac{1}{3}\ d\theta = \frac{2\pi}{3}.$$

2 (a) The region of integration is half the unit circle:

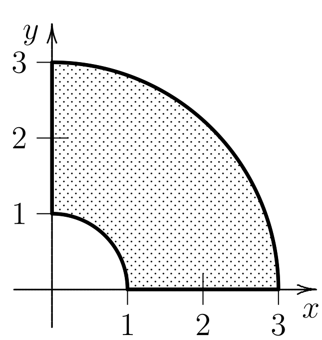


(b) In polar coordinates, the region is $-\pi/2 \le \theta \le \pi/2$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^1 r\cos\theta\ r\ dr\ d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos\theta\ dr\ d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3} \cos\theta\ d\theta = \frac{2}{3}.$$

3 (a) The region of integration is one quarter of an annulus (the region between two circles):



(b) In polar coordinates, the region is $0 \le \theta \le \pi/2$, $1 \le r \le 3$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{\pi/2} \int_1^3 (r\cos\theta + r\sin\theta)^2 \; r \; dr \; d\theta = \int_0^{\pi/2} \int_1^3 r(r\cos\theta + r\sin\theta)^2 \; dr \; d\theta.$$

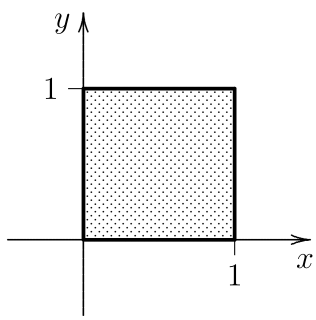
Multiplying out the square and using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, the integrand becomes

$$r(r\cos\theta + r\sin\theta)^2 = r^3\cos^2\theta + r^3\sin^2\theta + 2r^3\sin\theta\cos\theta = r^3(1 + \sin 2\theta).$$

Thus the integral is

$$\int_0^{\pi/2} \int_1^3 r^3(1 + \sin 2\theta) \; dr \; d\theta = \int_0^{\pi/2} 20(1 + \sin 2\theta) \; d\theta = 10\pi + 20.$$

4 (a) The region of integration is a square:

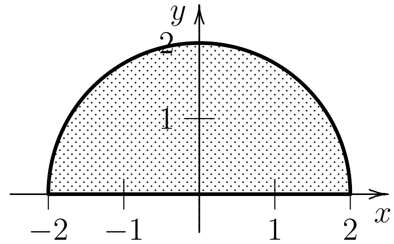


(b) This region is not easy to describe in polar coordinates! However, it's easy to describe in rectangular coordinates because it is just a square.

(c) In rectangular coordinates, the integral is

$$\int_0^1 \int_0^1 \left(x^2 + y^2\right) \; dx \; dy = \int_0^1 \left(\frac{1}{3} + y^2\right) \; dy = \frac{2}{3}.$$

5 (a) The region of integration is a half circle:



(b) In polar coordinates, the region is $0 \le \theta \le \pi$, $0 \le r \le 2$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^\pi \int_0^2 e^{r^2}\ r\ dr\ d\theta = \int_0^\pi \int_0^2 re^{r^2}\ dr\ d\theta.$$

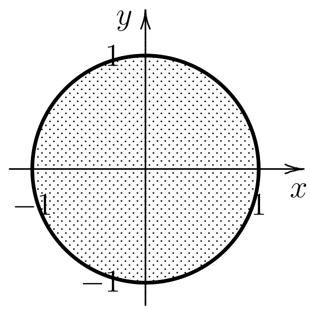
To evaluate the inner integral, set $u = r^2$. Then $du = 2rdr$ and u goes from 0 to 4 as r goes from 0 to 2, so

$$\int_0^2 re^{r^2}\ dr = \frac{1}{2}\int_0^4 e^u\ du = \frac{e^4-1}{2}.$$

Thus the double integral is

$$\int_0^\pi \frac{e^4-1}{2}\ d\theta = \pi\frac{e^4-1}{2}.$$

6 (a) The region of integration is the unit circle:



(b) In polar coordinates, the region is $0 \le \theta \le 2\pi$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^1 r\sqrt{1-r^2}\ dr\ d\theta.$$

To evaluate the inner integral, set $u = 1-r^2$. Then $du = -2rdr$ and u goes from 1 to 0 as r goes from 0 to 1, so

$$\int_0^1 r\sqrt{1-r^2}\ dr = -\frac{1}{2}\int_1^0 \sqrt{u}\ du = \frac{1}{3}.$$

Thus the double integral is

$$\int_0^{2\pi} \frac{1}{3}\ d\theta = \frac{2\pi}{3}.$$

Notice that geometrically, this integral is just calculating the volume of the upper hemisphere of $x^2+y^2+z^2 = 1$.

7 First, we must find the region to integrate over; this will just be the interior of the intersection of the paraboloid with the xy-plane (see Figure 1). That intersection is given by setting $z = 0$ to get $9-x^2-y^2 = 0$, or $x^2+y^2 = 9$. Thus we want to integrate over the disk $x^2+y^2 \le 9$. The

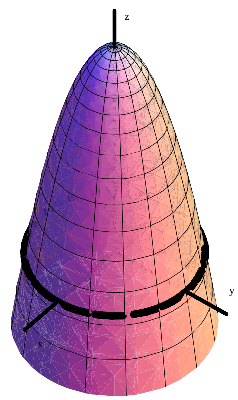


Figure 1: Surface for Problem 7

function to integrate is the height of the paraboloid above the xy-plane, which is $z = 9 - x^2 - y^2$. Thus we want to calculate the integral

$$\iint_R \left(9 - x^2 - y^2\right)\, dA$$

for R the disk $x^2 + y^2 \leq 9$. In polar coordinates, this integral is

$$\int_0^{2\pi}\int_0^3 (9 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left(9r - r^3\right)\, dr d\theta = \int_0^{2\pi} \left(\frac{81}{2} - \frac{81}{4}\right)\, d\theta = \frac{81\pi}{2}.$$

8 First, we must find the region to integrate over; this will just be the interior of the intersection of the sphere with the xy-plane (see picture). That intersection is given by setting $z = 0$ to get $x^2 + y^2 + 1 = 4$, or $x^2 + y^2 = 3$. Thus we want to integrate over the disk $x^2 + y^2 \leq 3$. The function to integrate is the height of the sphere above the xy-plane, which is $z = \sqrt{4 - x^2 - y^2} - 1$. Thus we want to calculate the integral

$$\iint_R \left(\sqrt{4 - x^2 - y^2} - 1\right)\, dA$$

for R the disk $x^2 + y^2 \leq 3$. In polar coordinates, this integral is

$$\int_0^{2\pi}\int_0^{\sqrt{3}} \left(\sqrt{4 - r^2} - 1\right)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{4 - r^2}\, dr\, d\theta - \int_0^{2\pi}\int_0^{\sqrt{3}} r\, dr\, d\theta.$$

The second term is just

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\, dr\, d\theta = \int_0^{2\pi} \frac{3}{2}\, d\theta = 3\pi.$$

For the inner integral in the first term, set $u = 4 - r^2$. Then $du = -2rdr$ and u goes from 4 to 1 as r goes from 0 to $\sqrt{3}$, so

$$\int_0^{\sqrt{3}} r\sqrt{4 - r^2}\; dr = -\frac{1}{2}\int_4^1 \sqrt{u}\; du = \frac{7}{3}.$$

Thus the first term is

$$\int_0^{2\pi} \frac{7}{3}\; d\theta = \frac{14\pi}{3}.$$

Putting it all together, the volume is

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{4 - r^2}\; dr\; d\theta - \int_0^{2\pi}\int_0^{\sqrt{3}} r\; dr\; d\theta = \frac{14\pi}{3} - 3\pi = \frac{5\pi}{3}.$$