Surface Area

1 In this problem, we'll find the surface area of a sphere of radius a. We think we know that the answer should be $4\pi a^2$, but now we've defined the area of a parameterized surface to be

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \; dA, \qquad (*)$$

so we should be able to make sure.

(a) Use spherical coordinates to write down a parameterization of the sphere of radius a. Recall that we can use spherical coordinates to parameterize the sphere of radius a as

$$\mathbf{r}(\phi, \theta) = \langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\rangle.$$

What values of ϕ and θ are needed to parameterize the entire sphere? (This will tell you the region R over which we will integrate.)

(b) Now find $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$ using the parameterization from part (a).

(c) Compute $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$. You should get $a^2\sin(\phi)$.

(d) Find the surface area of the sphere using the formula $(*)$.

2 One particular parameterization that we might take is for the graph of a function $z = f(x, y)$. We can then replace (u, v) with (x, y), so we get the parameterization

$$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle .$$

(a) Use the above expression for $\mathbf{r}(x, y)$ to compute $\mathbf{r}_x$, $\mathbf{r}_y$, and $|\mathbf{r}_x \times \mathbf{r}_y|$.

(b) Use your answer to part (a) and equation $(*)$ to find that the surface area of the graph of the function $f(x, y)$ is

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_R \sqrt{1 + |\nabla f|^2}\, dA. \qquad (**)$$

(c) Use equation $(**)$ to find the surface area of the paraboloid $z = x^2 + y^2$ that lies over the disk $x^2 + y^2 \leq 9$ of radius 3.

3 Another particularly straightforward set of examples is surfaces of revolution. If we revolve the graph a function $f(x)$ ($a \leq x \leq b$) around the x-axis, we get a surface that may be parameterized by

$$\mathbf{r}(x, \theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle .$$

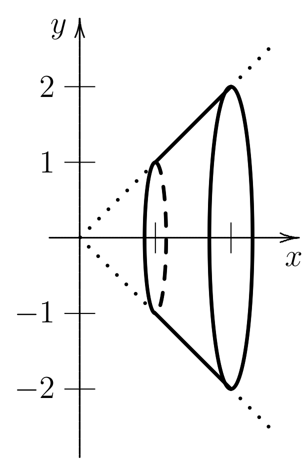
(a) Compute $\mathbf{r}_x$ and $\mathbf{r}_\theta$.

(b) Show that $|\mathbf{r}_x \times \mathbf{r}_\theta| = f(x)\sqrt{1 + (f'(x))^2}$.

(c) Use your answer to part (b) to deduce that the area of this surface of revolution is given by

$$A = 2\pi \int_a b f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (\dagger)$$

4 Use equation (†) to compute the area of the *frustrum* of a cone obtained by revolving the line $y = x$ (between $x = 1$ and $x = 2$) around the x-axis. (Note: this problem is really easy using the above equation.)



5 Let's re-do Problem 1, now thinking of the sphere as a surface of revolution. Compute the surface area of a sphere of radius a by revolving the curve $y = \sqrt{a^2 - x^2}$ ($-a \leq x \leq a$) around the x-axis.