Triple Integrals

1 Evaluate the following triple integrals as iterated integrals.

(a) $\iiint_E xy\, dV$, where $E = [0,1] \times [0,2] \times [0,3]$.

(b) $\iiint_E xy^3 z^2\, dV$, where $E = [-1,1] \times [-3,3] \times [0,3]$.

(c) $\iiint_E y^2 z \cos(xyz)\, dV$, where $E = [0,\pi] \times [0,1] \times [0,2]$.

Hint: Try using different orders of integration.

2 For each of the following regions E, write the triple integral $\iiint_E f(x,y,z)\, dV$ as an iterated integral. There may be up to six different ways to do this, depending on whether you write it with $dx\, dy\, dz$ or $dz\, dy\, dx$ or $dx\, dz\, dy$ or...

(a) The tetrahedron bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$.

(b) The (solid) sphere $x^2 + y^2 + z^2 = a^2$.

(c) The region between the paraboloid $x = 1 - y^2 - z^2$ and the yz-plane.

(d) The region bounded by the surface $z = 3xy + 1$ and the planes $z = 0$, $x = 0$, $x = 1$, $y = 0$, and $y = x$.

(e) The region bounded by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $y + z = 2$.

(f) The pyramid whose base is the square $[-1, 1] \times [-1, 1]$ in the xy-plane and whose vertex is the point $(0, 0, 1)$.

3 Evaluate the following integrals.

(a) $\iiint_E 1 \, dV$, where E is the region in Problem 2(a).

(b) $\iiint_E z \, dV$, where E is the region in Problem 2(d).

4 Find the volume of the pyramid described in Problem 2(f).

5 Consider a brick in the region $[0, 1] \times [0, 2] \times [0, 1]$ whose density at a point (x, y, z) is $\rho(x, y, z) = 2 + xy - 2z$. Find the mass of the brick.

Triple Integrals – Solutions

1 (a) The integral is

$$\int_0^1 \int_0^2 \int_0^3 xy\, dz\, dy\, dx = \int_0^1 \int_0^2 3xy\, dy\, dx$$
$$= \int_0^1 3x \left(\frac{2^2}{2} - 0 \right) dx$$
$$= \int_0^1 6x\, dx = 3.$$

(b) The integral is

$$\int_0^3 \int_{-3}^3 \int_{-1}^1 xy^3 z^2\, dx\, dy\, dz = \int_0^3 \int_{-3}^3 y^3 z^2 \left(\frac{1^2}{2} - \frac{(-1)^2}{2} \right) dy\, dz = 0.$$

Note that we can put the three integrals in whatever order we want, and putting the integral with respect to x first makes the computation easier.

(c) If we write the integral as

$$\int_0^\pi \int_0^1 \int_0^2 y^2 z \cos(xyz)\, dz\, dy\, dx,$$

we have to use integration by parts and the integral is lots of work. However, if we write it as

$$\int_0^1 \int_0^2 \int_0^\pi y^2 z \cos(xyz)\, dx\, dz\, dy,$$

it is a lot easier. In the inner integral, we can note that

$$\frac{\partial}{\partial x} y \sin(xyz) = y^2 z \cos(xyz)$$

so we get

$$\int_0^1 \int_0^2 \int_0^\pi y^2 z \cos(xyz)\, dx\, dz\, dy = \int_0^1 \int_0^2 (y \sin(\pi yz) - y \sin(0))\, dz\, dy$$
$$= \int_0^1 \int_0^2 y \sin(\pi yz)\, dz\, dy.$$

We then similarly have

$$\frac{\partial}{\partial z} \cos(xyz) = -y \sin(xyz)$$

so

$$\int_0^1 \int_0^2 y \sin(\pi yz)\, dz\, dy = \frac{-1}{\pi} \int_0^1 (\cos(2\pi y) - \cos(0))\, dy$$
$$= \frac{-1}{\pi} \int_0^1 (\cos(2\pi y) - 1)\, dy = \frac{1}{\pi}.$$

2 We only give one possible way to express each integral; there are others that are equally correct.

(a) $\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$

(b) $\iiint_E f(x,y,z)\,dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$

(c) This region is defined by the inequality $0 \le x \le 1 - y^2 - z^2$. For $1 - y^2 - z^2$ to be nonnegative, we also need (y,z) to lie on the disk D bounded by $y^2 + z^2 = 1$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dA$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dz\,dy.$$

(d) This region is defined by the inequalities $0 \le z \le 3xy + 1$, $0 \le x \le 1$, and $0 \le y \le x$, so we get

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^x \int_0^{3xy+1} f(x,y,z)\,dz\,dy\,dx.$$

(e) Being between the planes $y = 0$ and $y + z = 2$ says that $0 \le y \le 2 - z$, and being inside the cylinder says that (x,z) is in the disk D bounded by $x^2 + z^2 = 1$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_0^{2-z} f(x,y,z)\,y\,dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-z} f(x,y,z)\,dy\,dz\,dx.$$

(f) In the pyramid, z ranges from 0 at the base to 1 at the top. The cross-section of the pyramid given by a fixed value of z is the square $[-1+z, 1-z] \times [-1+z, 1-z]$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} f(x,y,z)\,dx\,dy\,dz.$$

3 (a) The integral is

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$
$$= \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2} \right) dx$$
$$= \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}.$$

(b) The integral is

$$\int_0^1 \int_0^x \int_0^{3xy+1} z \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{(3xy+1)^2}{2} \, dy \, dx$$
$$= \frac{1}{2} \int_0^1 \int_0^x (9x^2y^2 + 6xy + 1) \, dy \, dx$$
$$= \frac{1}{2} \int_0^1 (3x^5 + 3x^3 + x) \, dx = \frac{7}{8}.$$

4 The volume of a region E is given by the integral $\iiint_E 1 \, dV$. In this case, that integral is

$$\int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} 1 \, dx \, dy \, dz = \int_0^1 \int_{-1+z}^{1-z} (2-2z) \, dy \, dz$$
$$= \int_0^1 (2-2z)^2 \, dz = \frac{4}{3}.$$

5 The mass is given by integrating the density over the region, so the mass is

$$\int_0^1 \int_0^2 \int_0^1 (2 + xy - 2z) \, dz \, dy \, dx = \int_0^1 \int_0^2 (2 + xy - 1) \, dy \, dx$$
$$= \int_0^1 (4 + 2x - 2) \, dy \, dx$$
$$= 4 + 1 - 2 = 3.$$