Cylindrical & Spherical Coordinates

When we change from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) or spherical coordinates (ρ, θ, ϕ), integrals transform according to the rule

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

$\boxed{1}$ Using cylindrical coordinates, evaluate the integral $\iiint_E \sqrt{x^2 + y^2}\,dV$, where E is the solid in the first octant inside the cylinder $x^2 + y^2 = 16$ and below the plane $z = 3$.

$\boxed{2}$ Sketch the solid whose volume is given by the integral $\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r\,dz\,dr\,d\theta$, and evaluate the integral.

$\boxed{3}$ Use spherical coordinates to evaluate $\iiint_E z\,dV$, where E lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the quarter-space where $y \leq 0$ and $z \geq 0$.

$\boxed{4}$ Use spherical coordinates to set up a triple integral expressing the volume of the "ice-cream cone," which is the solid lying above the cone $\phi = \pi/4$ and below the sphere $\rho = \cos\phi$. Evaluate it.

$\boxed{5}$ Sketch the region of integration for

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx,$$

and evaluate the integral by changing to spherical coordinates.

$\boxed{6}$ Make an appropriate change of coordinates to evaluate the integral $\iiint_E (x^2 + y^2)\, dV$, where E is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.

Cylindrical & Spherical Coords. – Answers and Solutions

$\boxed{1}$ The solid E may be described as

$$\begin{aligned} E &= \left\{(x,y,z)\ :\ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 16, 0 \leq z \leq 3\right\} \\ &= \left\{(r,\theta,z)\ :\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq 4, 0 \leq z \leq 3\right\}. \end{aligned}$$

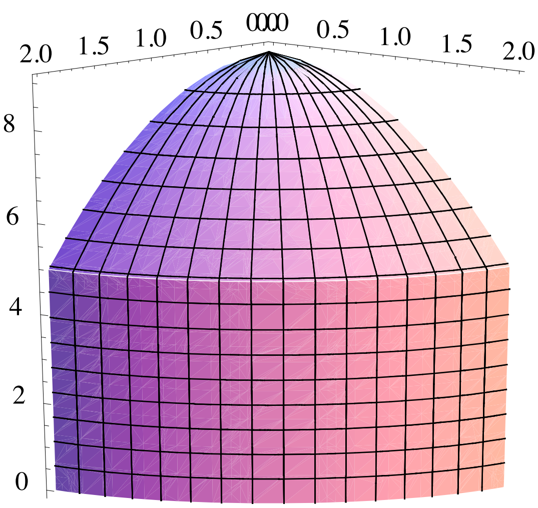
Thus it makes sense to evaluate this in cylindrical coordinates:

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^3 \int_0^{\pi/2} \int_0^4 r\ r\ dr\ d\theta\ dz = 3 \cdot \frac{\pi}{2} \cdot \frac{1}{3}(4)^3 = 32\pi.$$

$\boxed{2}$ This solid can be described as

$$\left\{(r,\theta,z)\ :\ 0 \leq z \leq 9-r^2,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq 2\right\}.$$

This is that part of the cylinder of radius 2 (it is centered along the z-axis and has equation $x^2+y^2=2^2$) that lies in the first octant and underneath the elliptic paraboloid $z = 9-x^2-y^2$. Here's a very simple Mathematica sketch of this solid:



$\boxed{3}$ In spherical coordinates, $z = \rho\cos(\phi)$. The region over which we're integrating can be described by the inequalities $0 \leq \phi \leq \frac{\pi}{2}$ (from $z \geq 0$), $\pi \leq \theta \leq 2\pi$ (from $y \leq 0$), and $1 \leq \rho \leq 2$ (from $1 \leq x^2+y^2+z^2 \leq 4$). Thus our integral is

$$\iiint_E z\,dV = \int_0^{\pi/2} \int_\pi^{2\pi} \int_1^2 \rho\cos(\phi) \cdot \rho^2 \sin(\phi) d\rho\ d\theta\ d\phi = \frac{15\pi}{8}.$$

$\boxed{4}$ This region is

$$\left\{(\rho,\theta,\phi)\ :\ 0 \leq \rho \leq \cos(\phi),\ 0 \leq \phi \leq \tfrac{\pi}{4},\ 0 \leq \theta \leq 2\pi\right\}.$$

Thus the volume of the "ice-cream cone" is expressed by the iterated integral

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos(\phi)} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta.$$

We were not asked to evaluate this, but it isn't difficult:

$$V = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\cos^3(\phi)\sin(\phi)\ d\phi\ d\theta = \int_0^{2\pi} \frac{1}{3} \cdot \frac{1}{4}\left(1^4 - \left(\frac{1}{\sqrt{2}}\right)^4\right) d\theta = \frac{\pi}{8}.$$

$\boxed{5}$ This is the region

$$\left\{(x,y,z)\ :\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2},\ 0 \le y \le \sqrt{1-x^2},\ 0 \le x \le 1\right\}.$$

The x and y restrictions mean we're integrating over the quarter of the unit circle in the first quadrant. The restrictions on z mean we're integrating the volume between the cone $z^2 = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$. In spherical coordinates, the cone is $\phi = \frac{\pi}{4}$ and the sphere is $\rho = \sqrt{2}$. Thus this is the region

$$\left\{(\rho,\phi,\theta)\ :\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le \phi \le \tfrac{\pi}{4},\ 0 \le \rho \le \sqrt{2}\right\}.$$

Thus (since $xy = \rho\sin(\phi)\cos(\theta)\cdot\rho\sin(\phi)\sin(\theta)$),

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx &= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^2\sin^2(\phi)\sin(\theta)\cos(\theta)\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi \\ &= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^4\sin^3(\phi)\sin(\theta)\cos(\theta)\ d\rho\ d\theta\ d\phi.\end{aligned}$$

These iterated integrals are each independent of the others, so this quantity is

$$\left(\int_0^{\pi/4}\sin^3(\phi)\ d\phi\right)\left(\int_0^{\pi/2}\sin(\theta)\cos(\theta)\right)\left(\int_0^{\sqrt{2}}\rho^4\ d\rho\right) = \left(2 - \frac{5}{2\sqrt{2}}\right)\cdot\frac{1}{2}\cdot\frac{4\sqrt{2}}{5} = \frac{1}{15}\left(4\sqrt{2}-5\right).$$

The second and third of these integrals are simple, and the first is not difficult using the substitution $u = \cos(\phi)$ and the relation $\sin^2(\phi) = 1 - \cos^2(\phi)$. We omit any further details.

$\boxed{6}$ We can use either spherical or cylindrical coordinates. Both have their appeal – the region (part of a sphere) calls out for spherical coordinates and the integrand ($r^2 = x^2 + y^2$) is asking for cylindrical. We'll do both.

In spherical coordinates, $x^2 + y^2 = \rho^2\sin^2(\phi)$ and the region E is simply

$$\left\{(\rho,\theta,\phi)\ :\ 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \tfrac{\pi}{2}\right\}.$$

Thus

$$\begin{aligned}\iiint_E (x^2+y^2)\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^2\sin^2(\phi)\cdot\rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^4\sin^3(\phi)\ d\rho\ d\theta\ d\phi.\end{aligned}$$

The only difficulty here is the integral of $\sin^3(\phi)$, but we computed this in problem 5. Hence the answer we get is

$$\begin{aligned}\iiint_E (x^2+y^2)\,dV &= \frac{1}{5}\int_0^{\pi/2}\int_0^{2\pi}\sin^3(\phi)\ d\theta\ d\phi \\ &= \frac{2\pi}{5}\int_0^{\pi/2}\sin^3(\phi)\ d\phi \\ &= \frac{4\pi}{15}.\end{aligned}$$

In cylindrical coordinates, $x^2 + y^2 = r^2$ and the region E is

$$\left\{(r, \theta, z) \;:\; 0 \leq z \leq \sqrt{1-r^2},\; 0 \leq r \leq 1,\; 0 \leq \theta \leq 2\pi\right\}.$$

(I'm angling to integrate z first, so I've written z in terms of r. One could integrate r first instead; in this case one would have $0 \leq r \leq \sqrt{1-z^2}$ and $0 \leq z \leq 1$). Thus

$$\iiint_E (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-z^2}} r^2 \cdot r \; dr \; dz \; d\theta.$$

These integrals are not at all complicated, and we get the same answer $\frac{4\pi}{15}$ as before.