MATH 21A

Vector Fields

SPRING, 2009

1 Match the following vector fields to the pictures, below. Explain your reasoning.

(Notice that in some of the pictures all of the vectors have been uniformly scaled so that the picture is more clear. Also notice that there are eight vector fields but only six pictures. There's probably a reason behind this.)

Here are the possible vector fields:

(a) $\mathbf{F}(x,y) = \langle 1, x \rangle$

(b) $\mathbf{F}(x,y) = \langle -y, x \rangle$

(c) $\mathbf{F}(x,y) = \langle y, x \rangle$

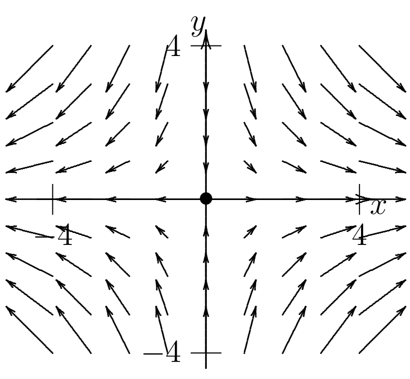
(d) $\mathbf{F}(x,y) = \langle 2x, -2y \rangle$

(e) ∇f, where $f(x,y) = x^2 + y^2$

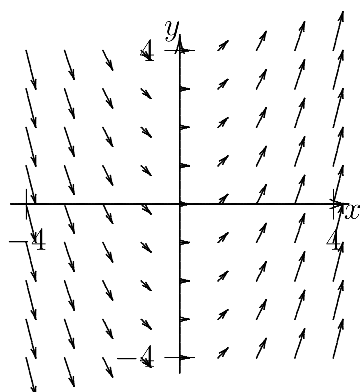
(f) ∇f, where $f(x,y) = \sqrt{x^2 + y^2}$

(g) ∇f, where $f(x,y) = xy$

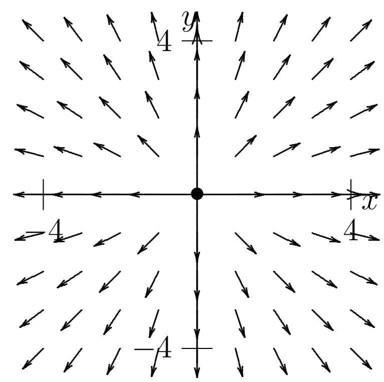
(h) ∇f, where $f(x,y) = x^2 - y^2$



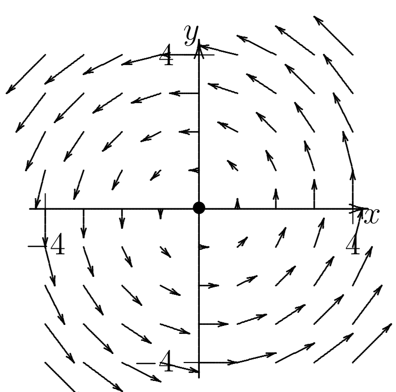
Field (I)



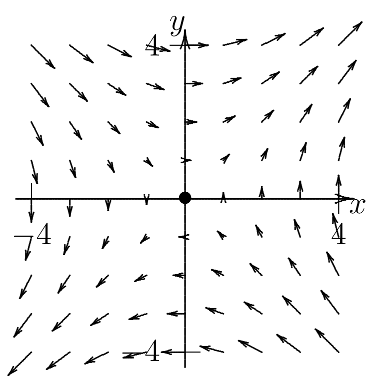
Field (II)



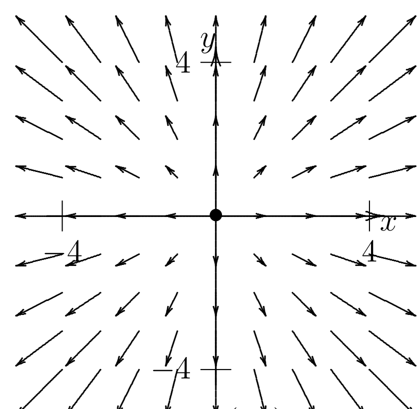
Field (III)



Field (IV)



Field (V)



Field (VI)

2 Recall that the gradient of a function is a vector normal to the level curve of this function. Explain how this confirms your identification of the pictures for vector fields (e) through (h), above.

Vector Fields – Answers and Solutions

1 (I) This is vector fields (d) and (h). (First notice that these two vector fields are the same!) We can see this by noticing that the vectors should point down when $y > 0$ and up when $y < 0$, and field (I) is the only one that does this.

(II) This is vector field (a). Notice that this vector field always has a positive rightward component, which is true only of Field (II).

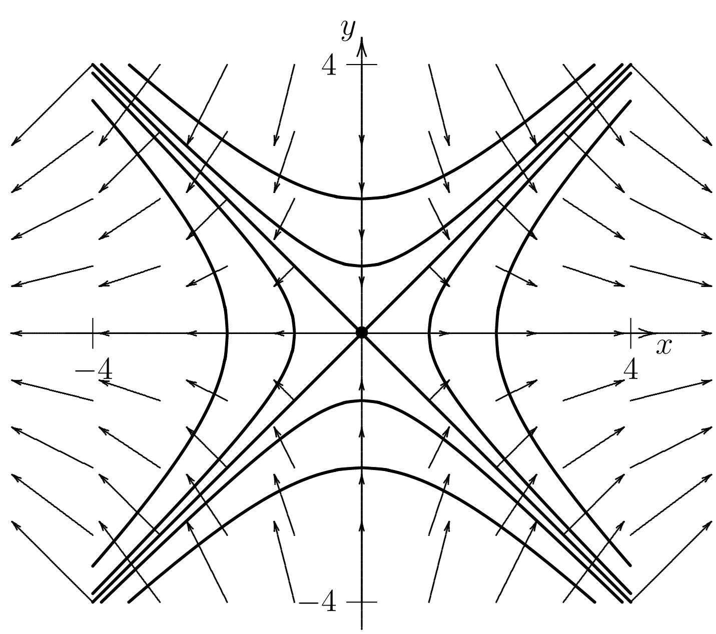
(III) This is vector field (f). Both (e) and (f) are vector fields that point radially outward, so they are Fields (III) and (VI). But which is which? Notice that the vector field in (e) is $\nabla f = \langle 2x, 2y \rangle$, which has length $2r = 2\sqrt{x^2 + y^2}$. On the other hand, the vector field in (f) is $\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle$, which has length 1. Thus (e) is Field (VI), the field with the vectors that increase in length as the distance from the origin increases, while (f) is Field (III), the field with vectors all the same magnitude.

(IV) This is vector field (b). Look, for example, at the vectors on the axes. On the x-axis, the vector field is $\mathbf{F}(x, 0) = \langle 0, x \rangle$, a vector that points vertically up (if $x > 0$) or down (if $x < 0$). This narrows our choices to Fields (IV) or (V). On the y-axis, the vector field is $\mathbf{F}(0, y) = \langle -y, 0 \rangle$, a vector that points to the left (if $y > 0$) or to the right (if $y < 0$). This eliminates Field (V) and confirms Field (IV).

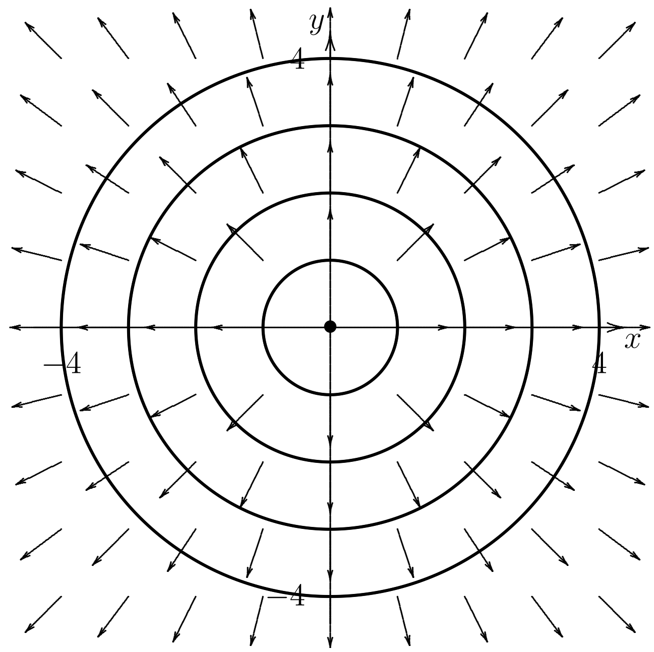
(V) This is vector field (c) and (g), by an analysis that is very similar to the one in Field (IV). (Notice that (c) and (g) are the same!)

(VI) This is vector field (e). See (III) for the explanation.

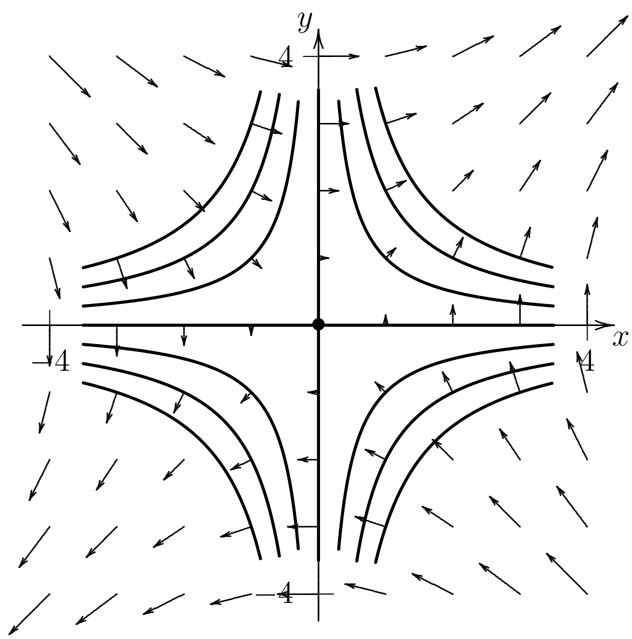
2 Four of the vector fields are (explicitly) gradient fields (we'll be able to tell later that vector fields (a) and (b) are *not* gradient fields). Since ∇f is perpendicular to the level curves of f, we should be able to see this in the vector field. Here I've re-drawn the four vector fields in question with some level curves drawn in as well – note the perpendicularity!



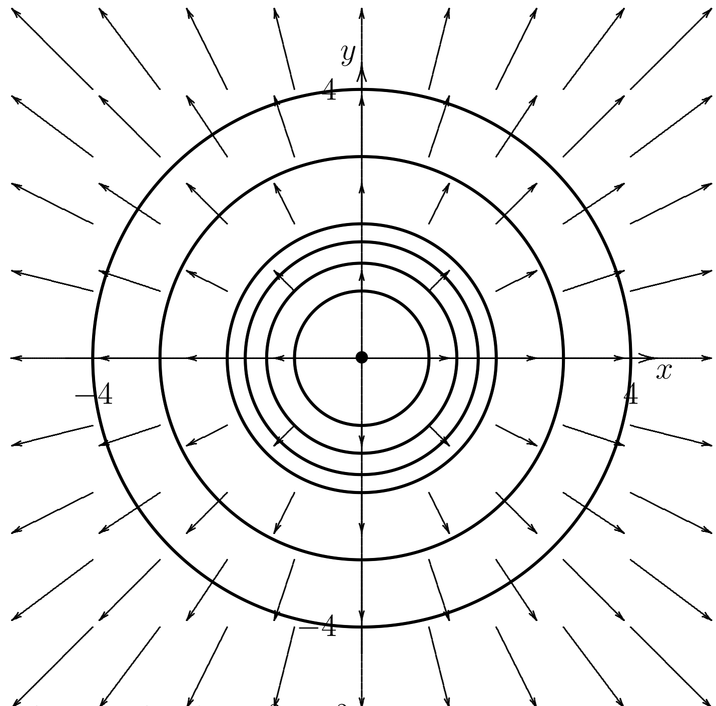
Field (I) and $f(x, y) = x^2 - y^2 = k$ for $k = 0, \pm 1, \pm 2$



Field (III) and $f(x,y) = \sqrt{x^2+y^2} = k$ with $k = 1, 2, 3, 4$



Field (V) with $f(x,y) = xy = k$ for $k = 0, \pm 1, \pm 2, \pm 3$



Field (VI) and $f(x, y) = x^2 + y^2 = k$ with $k = 1$, 2, 3, 4, 9, and 16