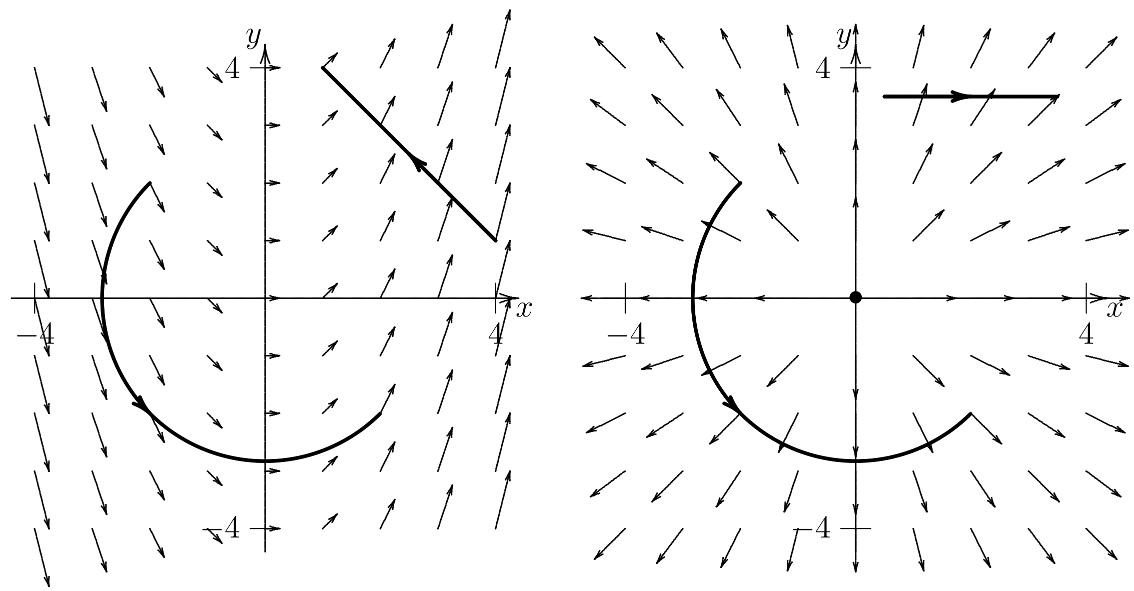
Line Integrals

1. Here are two of the vector fields from the earlier worksheet:



Field (II) Field (III)

For each curve C drawn on the vector field, determine if possible whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ would be positive, negative, or zero.

2. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following pairs of vector fields $\mathbf{F}$ and curves C.

(a) $\mathbf{F} = \langle y, x \rangle$ and C is the quarter-circle centered at the origin starting at $(2, 0)$ and proceeding counterclockwise to $(0, 2)$

(b) $\mathbf{F} = \langle y, x \rangle$ and C is the line segment starting at $(2, 0)$ and proceeding counterclockwise to $(0, 2)$

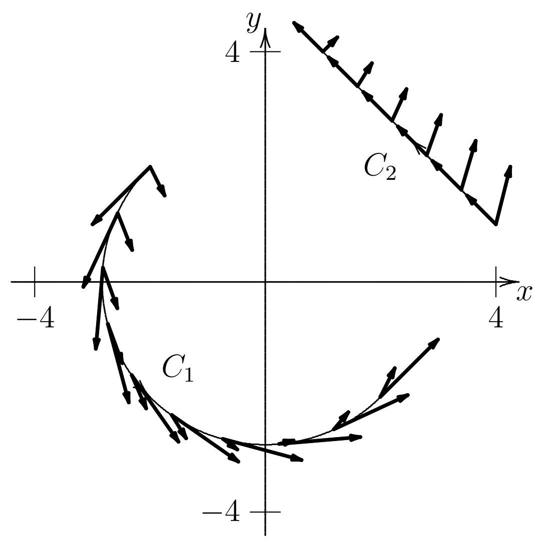
(c) $\mathbf{F} = \langle x, y, -2z \rangle$ and C is the "twisted cubic" $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = 1$.

(d) $\mathbf{F} = \langle x, y, -2z \rangle$ and C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$

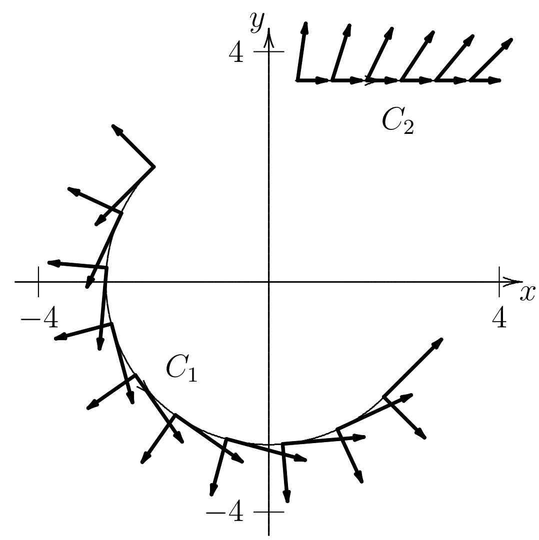
3. Suppose a spring is made by dipping a coil in the shape of the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $(0 \leq t \leq 2\pi)$ into molten steel. It turns out that the spring is thicker at the base then at the top as the liquid metal is influenced by gravity. We can model the mass of this spring by thinking of it as a curve with density $\delta(x, y, z) = 10 - z$. Find the mass of this spring.

Line Integrals – Solutions

1 Here's the vector field re-drawn so that only those vectors based at points on the curve remain. I've also drawn in tangent vectors at these same points, and I've sketched the curves with a thinner line:



Field (II)



Field (III)

Field (II) Notice that the angles between $\mathbf{F}$ and $\mathbf{r}'(t)$ are always acute (between 0° and 90°); thus $\mathbf{F} \cdot \mathbf{r}'(t) \geq 0$. This is true for both curve C_1 and curve C_2, so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} > 0$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} > 0$. I've made two leaps here: the first is that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt,$$

so rather than dotting $\mathbf{F}$ and "$d\mathbf{r}$" (whatever that is), we're looking at the dot product of $\mathbf{F}$ and $\mathbf{r}'(t)$. The other leap is that the integrals are positive, not just greater than or equal to zero. The point here is that we'd have to have $\mathbf{F} \cdot \mathbf{r}'(t) = 0$ (that is, $\mathbf{F}$ and $\mathbf{r}'(t)$ perpendicular) everywhere in order for the line integral to be zero (since the dot product is never negative).

Field (III) Once again the angles between $\mathbf{F}$ and $\mathbf{r}'(t)$ are always acute, but this time something interesting happens. On curve C_1, the angle between $\mathbf{F}$ and $\mathbf{r}'(t)$ is always 90°), so $\mathbf{F} \cdot \mathbf{r}'(t) = 0$ on C_1. Thus $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}0$ while $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} > 0$.

2 (a) This curve is part of a circle of radius 2, so we use the parameterization $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$. Since we're only interested in the quarter-circle in the first quadrant, we restrict t to $0 \leq t \leq \frac{\pi}{2}$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \langle 2\sin(t), 2\cos(t)\rangle \cdot \langle -2\sin(t), 2\cos(t)\rangle\, dt = 4\int_0^{\pi/2} \big(\cos^2(t) - \sin^2(t)\big)\, dt.$$

We'll use the formula $\cos(2t) = \cos^2(t) - \sin^2(t)$ to simplify this to

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4\int_0^{\pi/2} \cos(2t)\, dt = 4 \cdot \frac{1}{2}\sin(2t)\Big|_0^{\pi/2} = 2\big(\sin(\pi) - \sin(0)\big) = 0.$$

(b) We can parameterize our line segment C by $\mathbf{r}(t) = \langle 2, 0\rangle + t\langle 0-2, 2-0\rangle = \langle 2-2t, 2t\rangle$ for $0 \le t \le 1$. (This is a common trick: the line segment from $\mathbf{a}$ to $\mathbf{b}$ is $\mathbf{a} + t\,(\mathbf{b} - \mathbf{a})$, with $0 \le t \le 1$.) We then get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2t, 2-2t\rangle \cdot \langle -2, 2\rangle \; dt = \int_0^1 (4 - 8t) \; dt = \Big[4t - 4t^2\Big]_0^1 = 0.$$

(c) Now we're given our parameterization $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ (with $0 \le t \le 1$), so the computation is straightforward:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t, t^2, -2t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle \; dt = \int_0^1 \left(t + 2t^3 - 6t^5\right) \; dt$$
$$= \left[\frac{1}{2}t^2 + \frac{1}{2}t^4 - t^6\right]_0^1 = \frac{1}{2} + \frac{1}{2} - 1 = 0.$$

(d) Here the only trouble is parameterizing the line segment. We use the same trick as in part (b) to find $\mathbf{r}(t) = \langle 0, 0, 0\rangle + t\langle 1-0, 1-0, 1-0\rangle = \langle t, t, t\rangle$ (with $0 \le t \le 1$). Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t, t, -2t\rangle \cdot \langle 1, 1, 1\rangle \; dt = \int_0^1 (t + t - 2t) \; dt = 0.$$

3 Here I'm trying to get you to find

$$\text{Mass} = \int_C \delta(x, y, z) \; ds.$$

How could we do this? The idea is that the mass of a small piece (the ith small piece) of the wire is roughly the density $\delta(x_i^*, y_i^*, z_i^*)$ times the arc length δs_i of this piece. Hence the mass is approximately

$$\text{Mass} \approx \sum_{i=1}^{n} \delta(x_i^*, y_i^*, z_i^*) \; \Delta s_i.$$

In the limit we end up with the formula given.

To compute this integral, we recall that $\delta(x, y, z) = 10 - z$ and $ds = |\mathbf{r}'(t)| \; dt$. Here $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$, so $|\mathbf{r}'(t)| = \sqrt{2}$ and $\delta(\mathbf{r}(t)) = 10 - t$. Thus

$$\text{Mass} = \int_C \delta(x, y, z) \; ds = \int_0^{2\pi} (10 - t)\sqrt{2} \; dt = \sqrt{2}\left[10t - \frac{1}{2}t^2\right]_0^{2\pi} = \sqrt{2}\left(20\pi - 2\pi^2\right).$$