More On Line Integrals

Theorem: Suppose C is a piecewise smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a function (of two or three variables) for which ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

1 For each of the following functions f and curves (paths) C, calculate $\int_C \nabla f \cdot d\mathbf{r}$ using the theorem, above.

(a) $f(x,y) = x^2 - y$,
C = quarter circle from $(2,0)$ to $(0,2)$

(b) $f(x,y) = x + 2y$,
C = line segment from $(0,0)$ to $(2,3)$

(c) $f(x,y) = 2 - x - y$,
C = three-quarters of the circle from $(0,2)$ to $(2,0)$

(d) $f(x,y) = e^{2x-y}$,
C = any curve from $(0,0)$ to $(2,2)$

As you can see from Problem 1, it is fairly simple to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ when we know that $\mathbf{F} = \nabla f$. These gradient fields are known as *conservative* vector fields. So the natural questions are:

(a) How can we tell if $\mathbf{F}$ is conservative?

(b) If $\mathbf{F}$ is conservative, how can we find f so that $\mathbf{F} = \nabla f$?

The next few problems are meant to answer these questions.

2 Suppose $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \langle P, Q \rangle$ is a conservative vector field; that is, $\mathbf{F} = \nabla f$ for some function $f(x,y)$

(a) Write both P and Q in terms of derivatives of f.

(b) Write both $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in terms of derivatives of f. How do they relate? (Hint: Use Clairaut's theorem!)

It turns out we have the following theorem:

> **Theorem:** Suppose $\mathbf{F} = \langle P, Q \rangle$ is a vector field on an open simply-connected domain D. Suppose that P and Q have continuous first-order derivatives and that
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D.$$
>
> Then $\mathbf{F}$ is conservative.

(Notice that this is the converse of what you showed in the previous problem! That is, between Problem 2 and this theorem, we now know that, in the right circumstances, $\mathbf{F} = \langle P, Q \rangle$ is conservative exactly when $P_y = Q_x$.)

3 Use this theorem to determine which of the following vector fields are conservative:

(a) $\mathbf{F} = \langle y, x \rangle$

(b) $\mathbf{F} = \langle x, y \rangle$

(c) $\mathbf{F} = \langle x^2 y, 2x \rangle$

(d) $\mathbf{F} = \langle 2x \sin(y), x^2 \cos(y) \rangle$

(e) $\mathbf{F} = \langle 3x^2, x - 4y \rangle$

(f) $\mathbf{F} = \langle 2y^2 + e^{x-y}, 4xy - e^{x-y} + 2 \rangle$

4 For each conservative vector field $\mathbf{F} = \nabla f$ in the last problem, find the potential function f.

More on Line Integrals – Solutions

1

(a) $\int_C \nabla f \cdot d\mathbf{r} = f(0,2) - f(2,0) = -2 - 4 = -6$

(b) $\int_C \nabla f \cdot d\mathbf{r} = f(2,3) - f(0,0) = 8 - 0 = 8$

(c) $\int_C \nabla f \cdot d\mathbf{r} = f(2,0) - f(0,2) = 0 - 0 = 0$

(d) $\int_C \nabla f \cdot d\mathbf{r} = f(2,2) - f(0,0) = e^2 - 1$

2

(a) If $\mathbf{F} = \langle P, Q\rangle$ equals $\nabla f = \langle f_x, f_y\rangle$, then $P = f_x$ and $Q = f_y$.

(b) Since $P = f_x$, we have $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(f_x) = f_{xy}$. Similarly, $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(f_y) = f_{yx}$. By Clairaut's theorem, these are the same thing: $P_y = Q_x$.

3

(a) Here $P = y$ and $Q = x$, so $P_y = 1 = Q_x$, and so $\mathbf{F}$ is conservative.

(b) Here $P = x$ and $Q = y$, so $P_y = 0 = Q_x$, and so $\mathbf{F}$ is conservative.

(c) Here $P = x^2y$ and $Q = 2x$, so $P_y = x^2$ but $Q_x = 2$; thus $\mathbf{F}$ is **not** conservative.

(d) Here $P = 2x\sin(y)$ and $Q = x^2\cos(y)$, so $P_y = 2x\cos(y) = Q_x$, and so $\mathbf{F}$ is conservative.

(e) Here $P = 3x^2$ and $Q = x - 4y$, so $P_y = 0$ but $Q_x = 1$; thus $\mathbf{F}$ is **not** conservative.

(f) Here $P = 2y^2 + e^{x-y}$ and $Q = 4xy - e^{x-y} + 2$, so $P_y = 4y - e^{x-y} = Q_x$, and so $\mathbf{F}$ is conservative.

4

(a) Here $f_x = y$, so $f(x,y) = xy + g(y)$ for some function $g(y)$. Taking the derivative, we find $f_y = x + g'(y)$. This is supposed to be $f_y = x$, so $g'(y) = 0$. This means $g(y) = C$, so $f(x,y) = xy + C$.

(b) Here $f_x = x$, so $f(x,y) = \frac{1}{2}x^2 + g(y)$ for some function $g(y)$. This means that $f_y = g'(y) = y$, so $g(y) = \frac{1}{2}y^2 + C$. Thus $f(x,y) = \frac{1}{2}(x^2 + y^2) + C$.

(d) Here $f_x = 2x\sin(y)$, so $f(x,y) = x^2\sin(y) + g(y)$ for some function $g(y)$. This means that $f_y = x^2\cos(y) + g'(y) = x^2\cos(y)$, so $g'(y) = 0$. Thus $f(x,y) = x^2\sin(y) + C$.

(f) Here $f_x = 2y^2 + e^{x-y}$, so $f(x,y) = 2xy^2 + e^{x-y} + g(y)$ for some function $g(y)$. This means that $f_y = 4xy - e^{x-y} + g'(y) = 4xy - e^{x-y} + 2$, so $g'(y) = 2$. Thus $g(y) = 2y + C$ and so $f(x,y) = 2xy^2 + e^{x-y} + 2y + C$.