

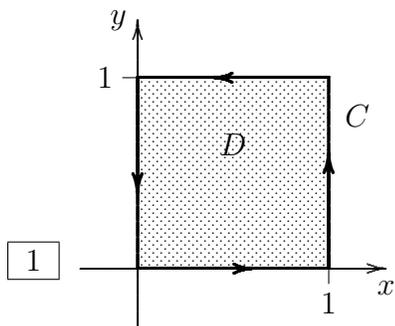
**Green's Theorem:** Suppose  $C$  is a positively oriented, piecewise-smooth, simple closed curve in the plane that bounds a region  $D$ . If  $P$  and  $Q$  have continuous derivatives (in an open set containing the region  $D$ ), then

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

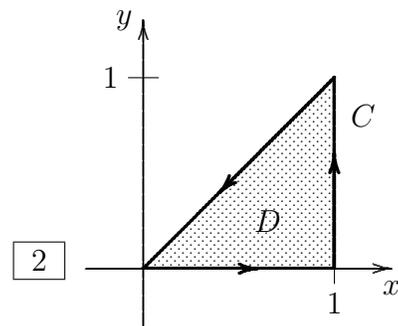
Sometimes the line integral is written  $\oint_C P dx + Q dy$  to emphasize that the curve is closed.

For each of the following regions  $D$ , associated boundary curves  $C$ , and line integrals...

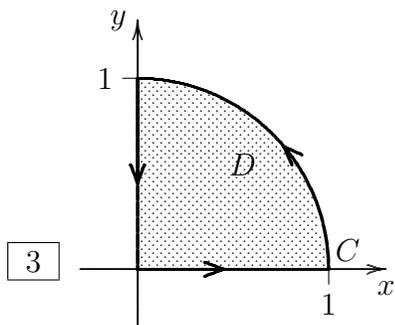
- (a) Compute the given line integral directly by parameterizing the path  $C$ .
- (b) Compute the given line integral by applying Green's theorem and computing a double integral.



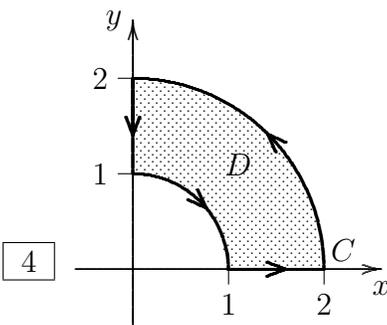
$$\int_C xy dx + (x^2 - y^2) dy$$



$$\int_C x^3 dx - xy^2 dy$$



$$\int_C x dy - y dx$$



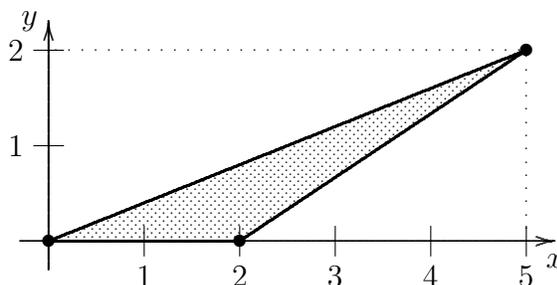
$$\int_C xy^2 dx - x^2y dy$$

5 How does this relate to our work on Friday? If the vector field  $\mathbf{F} = \langle P, Q \rangle$  is conservative (that is, if  $\mathbf{F} = \nabla f$  for some  $f$ ), what does Green's theorem say about the line integral and the double integral?

6 Some trickery: Recall that  $\iint_D 1 \, dA$  is the area of the region  $D$ . So if we choose  $P$  and  $Q$  so that  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ , then  $\int_C P \, dx + Q \, dy$  is also the area of  $D$ .

(a) One such choice is  $P = -y$  and  $Q = 0$ . Write down two other choices of the vector field  $\mathbf{F} = \langle P, Q \rangle$  so that  $\int_C P \, dx + Q \, dy$  is the area of  $D$ .

(b) Use one of your line integrals from part (a) to find the area of the triangle with corners at  $(0, 0)$ ,  $(2, 0)$ , and  $(5, 2)$ :



7 In this problem we'll calculate

$$\oint_C \frac{x \, dy - y \, dx}{x^2 + y^2}$$

where  $C$  is *any* positively oriented simple closed curve that encloses the origin, as follows:

(a) Let  $C_1$  and  $C_2$  be two different such simple closed curves that don't intersect. Let  $D$  be the region with boundary  $C_1 \cup (-C_2)$ . Show that

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = 0.$$

(b) Since

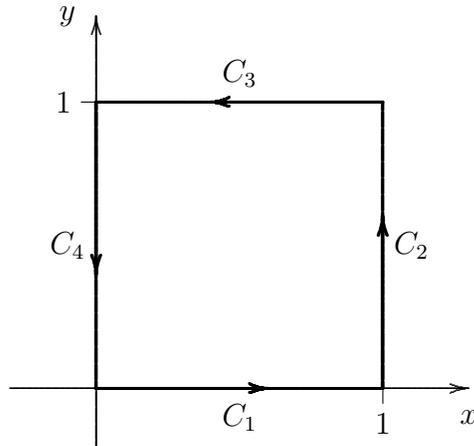
$$\int_C P \, dx + Q \, dy = \int_{C_1} P \, dx + Q \, dy + \int_{-C_2} P \, dx + Q \, dy = \int_{C_1} P \, dx + Q \, dy - \int_{C_2} P \, dx + Q \, dy,$$

use Green's theorem to show that the two line integrals (over  $C_1$  and  $C_2$ ) must agree.

(c) Choose a particularly nice parameterized curve (say, a simple circle) to compute the given integral.

## Green's Theorem – Answers and Solutions

- 1 (a) I'll write this line integral as a sum of four line integrals, each of which I'll parameterize and compute separately:



We perform the parameterizations without comment:

$C_1$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = dt$  and  $dy = 0$ . Thus

$$\int_{C_1} xy \, dx + (x^2 - y^2) \, dy = \int_0^1 t \cdot 0 \, dt + (t^2 - 0^2) \cdot 0 = \int_0^1 0 = 0.$$

$C_2$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 1, t \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = 0$  and  $dy = dt$ . Thus

$$\int_{C_2} xy \, dx + (x^2 - y^2) \, dy = \int_0^1 1 \cdot t \cdot 0 + (1^2 - t^2) \, dt = \int_0^1 (1 - t^2) \, dt = \frac{2}{3}.$$

$C_3$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 1 - t, 1 \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = -dt$  and  $dy = 0$ . Thus

$$\int_{C_3} xy \, dx + (x^2 - y^2) \, dy = \int_0^1 (1 - t) \cdot 1 \cdot -dt + ((1 - t)^2 - 1^2) \cdot 0 = \int_0^1 (t - 1) \, dt = -\frac{1}{2}$$

$C_4$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 0, 1 - t \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = 0$  and  $dy = -dt$ . Thus

$$\int_{C_4} xy \, dx + (x^2 - y^2) \, dy = \int_0^1 0 \cdot (1 - t) \cdot 0 + (0^2 - (1 - t)^2) \cdot -dt = \int_0^1 (1 - t)^2 \, dt = \frac{1}{3}.$$

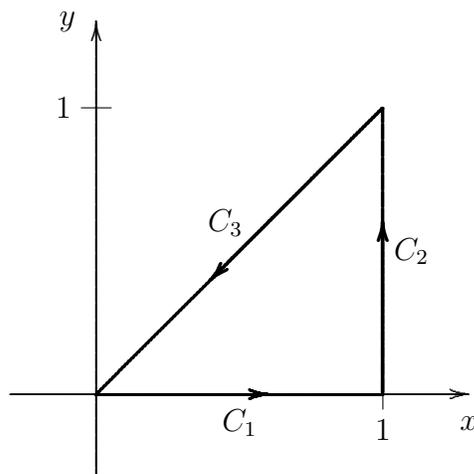
Putting this all together, we get

$$\int_C xy \, dx + (x^2 - y^2) \, dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + \frac{2}{3} - \frac{1}{2} + \frac{1}{3} = \frac{1}{2}.$$

- (b) Here  $P = xy$  and  $Q = x^2 - y^2$ , so the integrand of the double integral is  $Q_x - P_y = 2x - x = x$ . The region of integration is a square, so the limits of integration are simple:

$$\int_C xy \, dx + (x^2 - y^2) \, dy = \iint_D x \, dA = \int_0^1 \int_0^1 x \, dy \, dx = \frac{1}{2}.$$

2 (a) Again we'll break the line integral into simple line segments:



Again we parameterize and integrate each piece more or less without commentary:

$C_1$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle$  for  $0 \leq t \leq 1$ , so  $dx = dt$  and  $dy = 0$ . Thus

$$\int_{C_1} x^3 dx - xy^2 dy = \int_0^1 t^3 dt - t(0)^2 \cdot 0 = \int_0^1 t^3 dt = \frac{1}{4}.$$

$C_2$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 1, t \rangle$  for  $0 \leq t \leq 1$ , so  $dx = 0$  and  $dy = dt$ . Thus

$$\int_{C_2} x^3 dx - xy^2 dy = \int_0^1 1^3 \cdot 0 - 1(t)^2 dt = \int_0^1 -t^2 dt = -\frac{1}{3}.$$

$C_3$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 1, 1 \rangle + t(\langle 0, 0 \rangle - \langle 1, 1 \rangle) = \langle 1 - t, 1 - t \rangle$  for  $0 \leq t \leq 1$ , so  $dx = -dt$  and  $dy = -dt$ . Thus

$$\int_{C_3} x^3 dx - xy^2 dy = \int_0^1 (1 - t)^3 \cdot -dt - (1 - t)(1 - t)^2 \cdot -dt = \int_0^1 0 = 0.$$

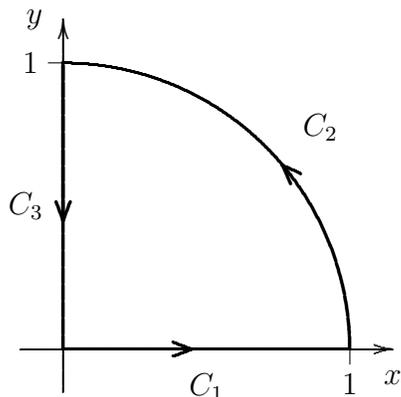
Putting this all together, we get

$$\int_C x^3 dx - xy^2 dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{4} - \frac{1}{3} + 0 = -\frac{1}{12}.$$

(b) Here  $P = x^3$  and  $Q = -xy^2$ , so the integrand of the double integral is  $Q_x - P_y = -y^2 - 0 = -y^2$ . The region of integration is a triangle, so the limits of integration are reasonably straightforward:

$$\int_C x^3 dx - xy^2 dy = \iint_D -y^2 dA = \int_0^1 \int_0^x -y^2 dy dx = \int_0^1 -\frac{1}{3}x^3 dx = -\frac{1}{12}.$$

- 3 (a) We'll break the line integral into two line segments and one arc:



Now we parameterize each part of the path and integrate, more or less without commentary:

$C_1$ :  $\mathbf{r}(t) = \langle t, 0 \rangle$  ( $0 \leq t \leq 1$ ), so  $dx = dt$ ,  $dy = 0$ , and  $x dy - y dx = t \cdot 0 - 0 dt = 0$ . Thus  $\int_{C_1} x dy - y dx = 0$ .

$C_2$ :  $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$  ( $0 \leq t \leq \frac{\pi}{2}$ ), so  $\langle dx, dy \rangle = \langle -\sin(t), \cos(t) \rangle dt$  and thus  $x dy - y dx = (\cos^2(t) + \sin^2(t)) dt = dt$ . Thus  $\int_{C_2} x dy - y dx = \int_0^{\pi/2} dt = \frac{\pi}{2}$ .

$C_3$ :  $\mathbf{r}(t) = \langle 0, 1 - t \rangle$  ( $0 \leq t \leq 1$ ), so  $dx = 0$ ,  $dy = -dt$ , and  $x dy - y dx = 0(-dt) - (1 - t)(0) = 0$ . Thus  $\int_{C_3} x dy - y dx = 0$ .

Putting this all together, we get

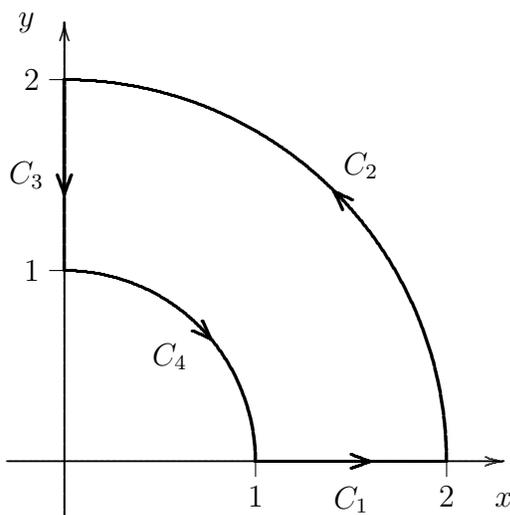
$$\int_C x dy - y dx = \int_{C_1} x dy - y dx + \int_{C_2} x dy - y dx + \int_{C_3} x dy - y dx = 0 + \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

- (b) Here  $P = -y$  and  $Q = x$ , so  $Q_x - P_y = 1 - (-1) = 2$ . Thus Green's theorem implies that

$$\int_C x dy - y dx = \iint_D 2 dA = 2 \cdot \text{Area}(D) = \frac{\pi}{2},$$

as before.

4 (a) As usual, we'll break the line integral into simple line segments and arcs:



Again we parameterize and integrate each piece more or less without commentary:

$C_1$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle$  for  $1 \leq t \leq 2$ , so  $dx = dt$  and  $dy = 0$ . Thus

$$\int_{C_1} xy^2 dx - x^2y dy = \int_1^2 t(0)^2 dt - t^2 \cdot 0 \cdot 0 = 0.$$

$C_2$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 2 \cos(t), 2 \sin(t) \rangle$  for  $0 \leq t \leq \frac{\pi}{2}$ , so  $dx = -2 \sin(t) dt$  and  $dy = 2 \cos(t) dt$ . Thus

$$\begin{aligned} \int_{C_2} xy^2 dx - x^2y dy &= \int_0^{\pi/2} 2 \cos(t) (2 \sin(t))^2 \cdot -2 \sin(t) dt - (2 \cos(t))^2 2 \sin(t) \cdot 2 \cos(t) dt \\ &= -16 \int_0^{\pi/2} (\sin^3(t) \cos(t) + \cos^3(t) \sin(t)) dt \\ &= -16 \int_0^{\pi/2} \sin(t) \cos(t) dt \quad (\text{since } \sin^2(t) + \cos^2(t) = 1) \\ &= -8. \end{aligned}$$

$C_3$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 0, 2 \rangle + t(\langle 0, 2 \rangle - \langle 0, 1 \rangle) = \langle 0, 2 - t \rangle$  for  $0 \leq t \leq 1$ , so  $dx = 0$  and  $dy = -dt$ . Thus

$$\int_{C_3} xy^2 dx - x^2y dy = \int_0^1 0(1-t)^2 \cdot 0 - 0^2(2-t) \cdot -dt = \int_0^1 0 = 0.$$

$-C_4$ : (We'll parameterize  $-C_4$  instead.)  $\mathbf{r}(t) = \langle x, y \rangle = \langle \cos(t), \sin(t) \rangle$  for  $0 \leq t \leq \frac{\pi}{2}$ , so  $dx = -\sin(t) dt$  and  $dy = \cos(t) dt$ . Thus

$$\begin{aligned} \int_{-C_4} xy^2 dx - x^2y dy &= \int_0^{\pi/2} \cos(t) (\sin(t))^2 \cdot -\sin(t) dt - (\cos(t))^2 \sin(t) \cdot \cos(t) dt \\ &= - \int_0^{\pi/2} (\sin^3(t) \cos(t) + \cos^3(t) \sin(t)) dt \\ &= - \int_0^{\pi/2} \sin(t) \cos(t) dt \quad (\text{since } \sin^2(t) + \cos^2(t) = 1) \\ &= -\frac{1}{2}. \end{aligned}$$

Putting this all together, we get

$$\int_C xy^2 dx - x^2y dy = \int_{C_1} + \int_{C_2} + \int_{C_3} - \int_{-C_4} = 0 + (-8) + 0 - \left(-\frac{1}{2}\right) = -\frac{15}{2}.$$

(b) Here  $P = xy^2$  and  $Q = -x^2y$ , so the integrand of the double integral is  $Q_x - P_y = -2xy - 2xy = -4xy$ . The region of integration is suited to polar coordinates, in which the limits of integration are simple:

$$\begin{aligned} \int_C xy^2 dx - x^2y dy &= \iint_D -4xy dA \\ &= \int_0^{\pi/2} \int_1^2 -4r^2 \sin(\theta) \cos(\theta) \cdot r dr d\theta \\ &= -4 \int_0^{\pi/2} \int_1^2 r^3 \sin(\theta) \cos(\theta) dr d\theta \\ &= -4 \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_1^2 \sin(\theta) \cos(\theta) d\theta \\ &= -15 \int_0^{\pi/2} \sin(\theta) \cos(\theta) d\theta \\ &= -15 \cdot \frac{1}{2} \sin^2(\theta) \Big|_0^{\pi/2} \\ &= -\frac{15}{2}. \end{aligned}$$

5 If  $\mathbf{F} = \langle P, Q \rangle$  is conservative, then we saw last time that  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ . This means that Green's theorem says that

$$\int_C P dx + Q dy = \iint_D 0 dA = 0.$$

Thus the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any simple closed curve  $C$ , provided the hypotheses of Green's theorem are satisfied. (In particular,  $P$  and  $Q$  must have continuous derivatives in an open set containing  $D$ , the region bounded by  $C$ . This is why Problem 7 isn't a contradiction, despite the fact that the vector field in that problem is  $\mathbf{F} = \nabla(\arctan(\frac{y}{x}))$ . Neither  $P$  nor  $Q$ , or even their derivatives are not continuous at the origin, which is part of the region bounded by each curve  $C$ .)

- 6 (a) There are lots of choices for  $P$  and  $Q$  that will result in  $Q_x - P_y = 1$ . For example, if  $P = ay$  and  $Q = bx$ , then  $Q_x - P_y = b - a$ , so there are many choices. Three typical ones are  $(a, b) = (-1, 0)$  (the given  $\langle P, Q \rangle = \langle -y, 0 \rangle$  of the problem),  $(a, b) = (0, 1)$  (so  $\langle P, Q \rangle = \langle 0, x \rangle$ ) and  $(a, b) = (\frac{1}{2}, \frac{1}{2})$ . These three examples give us the following line integrals that are equal to the area of  $D$ :

$$\int_C -y \, dx \quad \text{and} \quad \int_C x \, dy \quad \text{and} \quad \frac{1}{2} \int_C x \, dy - y \, dx.$$

While these are the traditional integrals, one could contrive much more complicated examples. For example, if

$$\begin{aligned} P(x, y) &= y(f'(x) + 1) + g(x) + h(y) \\ Q(x, y) &= f(x) + h'(y)x + k(y) \end{aligned}$$

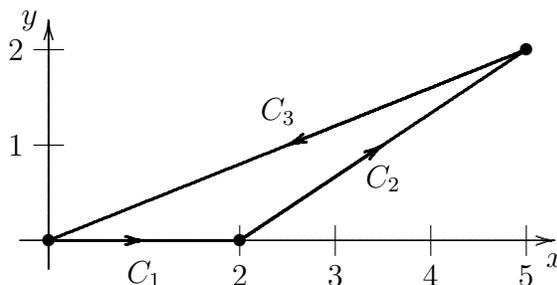
(where  $f(x)$ ,  $g(x)$ ,  $h(y)$  and  $k(y)$  are any continuously twice-differentiable functions), then

$$\frac{\partial P}{\partial y} = f'(x) + 1 + 0 + h'(y) \quad \text{and} \quad \frac{\partial Q}{\partial x} = f'(x) + h'(y) + 0,$$

so  $P_y - Q_x = 1$ . Thus

$$\oint_C \left( y(f'(x) + 1) + g(x) + h(y) \right) dx + \left( f(x) + h'(y)x + k(y) \right) dy = \text{Area}(D).$$

- (b) Let's use one of the simpler integrals – say,  $\int_C x \, dy$  – to find the area of this triangle. We'll label the three sides and compute the three line integrals, as usual, without much commentary:



$C_1$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle$  (for  $0 \leq t \leq 2$ ), so  $dx = dt$  and  $dy = 0$ . Thus

$$\int_{C_1} x \, dy = \int_0^2 t \cdot 0 = 0.$$

$C_2$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 2, 0 \rangle + t\langle 3, 2 \rangle = \langle 2 + 3t, 2t \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = 3 \, dt$  and  $dy = 2 \, dt$ . Thus

$$\int_{C_2} x \, dy = \int_0^1 (2 + 3t) \cdot 2 \, dt = 7.$$

$C_3$ :  $\mathbf{r}(t) = \langle x, y \rangle = \langle 5, 2 \rangle + t\langle -5, -2 \rangle = \langle 5 - 5t, 2 - 2t \rangle$  (for  $0 \leq t \leq 1$ ), so  $dx = -5 \, dt$  and  $dy = -2 \, dt$ . Thus

$$\int_{C_3} x \, dy = \int_0^1 (5 - 5t) \cdot -2 \, dt = -5.$$

Putting this all together, we get

$$\text{Area}(D) = \int_C y \, dx = \int_{C_1} y \, dx + \int_{C_2} y \, dx + \int_{C_3} y \, dx = 0 + 7 - 5 = 2.$$

7 (a) Here  $P = -\frac{y}{x^2+y^2}$  and  $Q = \frac{x}{x^2+y^2}$ , so

$$\frac{\partial P}{\partial y} = -\frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial Q}{\partial x} = \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ , so

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$$

(b) We've already seen that the double integral in Green's theorem must be zero, so we must have

$$\int_{C_1} P dx + Q dy - \int_{C_2} P dx + Q dy = 0$$

or that the two integrals are equal.

This now implies that the integrals are the same for *every* such simple closed curves, even pairs that intersect. If  $C_1$  and  $C_2$  intersect, then choose a third curve  $C_3$  (for example a really large or really small circle) that intersects neither curve. The above argument shows that the integrals over  $C_1$  and  $C_2$  both equal the integral over  $C_3$ , so all three must be the same!

(c) Let  $C$  be the unit circle, parameterized by  $\mathbf{r}(t) = \langle x, y \rangle = \langle \cos(t), \sin(t) \rangle$  (for  $0 \leq t \leq 2\pi$ ). Then  $dx = -\sin(t) dt$  and  $dy = \cos(t) dt$ , so

$$\begin{aligned} \int_C \frac{x dy - y dx}{x^2 + y^2} &= \int_0^{2\pi} \frac{\cos^2(t) dt + \sin^2(t) dt}{1} \\ &= \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

Thus  $\oint_C \frac{x dy - y dx}{x^2 + y^2} = 2\pi$  for *any* positively oriented simple closed curve that encloses the origin.

**Comment:** The interesting thing about this example is that  $\mathbf{F}$  is a gradient field:

$$\mathbf{F} = \nabla (\arctan(y/x)) = \left\langle \frac{-y/x^2}{1 + (y/x)^2}, \frac{1/x}{1 + (y/x)^2} \right\rangle = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

But the domain  $D$  of  $\mathbf{F}$  is the entire plane *except* for the origin. Thus  $D$  is open but not simply connected (it has a hole!), so being a gradient field is not the same thing as integrals being independent of path on  $D$ . (This is worked out in detail in the book as well; see example 13.4.5 on page 937).