Flux Integrals

A surface integral is

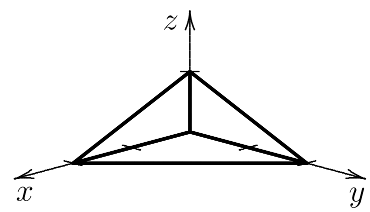
$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v))\ |\mathbf{r}_u \times \mathbf{r}_v|\ du\, dv,$$

where f is a function defined on the parametric surface $\mathbf{r}(u,v)$.

$\boxed{1}$ Evaluate the surface integral

$$\iint_S (1+z)\ dS,$$

where S is that part of the plane $x+y+2z=2$ in the first octant.



Suppose $\mathbf{F}$ is a continuous vector field on an oriented surface S with unit normal vector $\mathbf{n}$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\ du\ dv$$

for a parametrically defined surface.

$\boxed{2}$ Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant with inward orientation.

$\boxed{3}$ Evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, 2z \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit square $[0,1] \times [0,1]$ with the *downward* orientation.

$\boxed{4}$ Evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2x + 2y)\mathbf{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit disk (centered at the origin) with upward orientation.

$\boxed{5}$ Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle -z, x, y \rangle$ and S is the full unit hemisphere (including the base!) on and above the xy-plane (so $x^2 + y^2 + z^2 = 1$ plus a disk) with the outward orientation.

Flux Integrals – Answers and Solutions

$\boxed{1}$ Here we use the parameterization $\mathbf{r}(x,y) = \langle x, y, 1 - \frac{1}{2}x - \frac{1}{2}y\rangle$. From this we find that

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \left\langle \tfrac{1}{2},\ \tfrac{1}{2},\ 1 \right\rangle.$$

Therefore

$$\iint_S (1+z)\ dS = \iint_D \left[1 + \left(1 - \frac{1}{2}x - \frac{1}{2}y\right)\right]\ dA.$$

The region D in our parameter space (the xy-plane) need to cover our surface is the triangle

$$D = \{(x,y)\ :\ x \geq 0,\ y \geq 0,\ x + y \leq 2\},$$

so we can write our limits as follows:

$$\begin{aligned}\iint_S (1+z)\ dS &= \int_0^2 \int_0^{2-x} \left(2 - \frac{1}{2}x - \frac{1}{2}y\right)\ dy\ dx \\ &= \int_0^2 \left(\frac{1}{4}x^2 - 2x + 3\right)\ dx \\ &= \frac{1}{12} \cdot 2^3 - 2^2 + 3 \cdot 2 = \frac{8}{3}.\end{aligned}$$

$\boxed{2}$ This is based on Problem 23 from Section 13.6 of the textbook.

One common parameterization of the sphere of radius a is simply using spherical coordinates (with $\rho = a$):

$$\mathbf{r}(\phi,\theta) = \big\langle a\sin(\phi)\cos(\theta), a\sin(\phi)\cos(\theta), a\cos(\phi)\big\rangle.$$

One could then compute $\mathbf{r}_\phi \times \mathbf{r}_\theta$, but it is easier to just remember that we've done this before and the answer is:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin(\phi)\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\cos(\theta), \cos(\phi)\big\rangle.$$

(See, for example, page 869 of the text.) The coefficient in front is simply the coefficient $\rho^2\sin(\phi)$ from the spherical volume element (with $\rho = a$) while the vector is simply the unit vector in the direction $\langle x, y, z\rangle$ (the radial vector, which is perpendicular to the tangent plane to the sphere).

Notice that the orientation is specified to be the *inward* normal, so we actually want $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

In any case, with this we proceed. In spherical coordinates (with $\rho = a = 2$ in this case), our vector field is

$$\mathbf{F} = \langle y, -x, z\rangle = \langle 2\sin(\phi)\sin(\theta), -2\sin(\phi)\cos(\theta), 2\cos(\phi)\rangle.$$

Thus

$$\begin{aligned}
&\mathbf{F}\cdot(\mathbf{r}_\theta\times\mathbf{r}_\phi)\\
&= -4\sin(\phi)\big\langle 2\sin(\phi)\sin(\theta), -2\sin(\phi)\cos(\theta), 2\cos(\phi)\big\rangle\cdot\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\cos(\theta), \cos(\phi)\big\rangle\\
&= -8\sin(\phi)\cos^2(\phi).
\end{aligned}$$

We integrate this over the domain $\{(\phi,\theta)\ :\ 0\le\phi\le\pi/2,\ 0\le\theta\le\pi/2\}$, so we have

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{\pi/2}\int_0^{\pi/2} -8\sin(\phi)\cos^2(\phi)\,d\theta\,d\phi\\
&= -4\pi\int_0^{\pi/2}\sin(\phi)\cos^2(\phi)\,d\phi\\
&= -4\pi\left[-\frac{1}{3}\cos^3(\phi)\right]_0^{\pi/2} = -\frac{4\pi}{3}.
\end{aligned}$$

Another approach is to use the parameterization by x and y

$$\mathbf{r}(x,y)=\left\langle x, y, \sqrt{4-x^2-y^2}\right\rangle$$

then integrating over the quarter circle in the xy-plane. Let's see how this goes:

$$\mathbf{r}_y\times\mathbf{r}_x=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & -\frac{y}{\sqrt{4-x^2-y^2}}\\ 1 & 0 & -\frac{x}{\sqrt{4-x^2-y^2}}\end{vmatrix}=\left\langle -\frac{x}{\sqrt{4-x^2-y^2}},\ -\frac{y}{\sqrt{4-x^2-y^2}},\ -1\right\rangle.$$

(Notice we've used $\mathbf{r}_y\times\mathbf{r}_x$ to get the inward-pointing normal.) Thus

$$\begin{aligned}
\mathbf{F}\cdot(\mathbf{r}_y\times\mathbf{r}_x) &= \langle y,-x,z\rangle\cdot\left\langle -\frac{x}{\sqrt{4-x^2-y^2}},\ -\frac{y}{\sqrt{4-x^2-y^2}},\ -1\right\rangle\\
&= \left\langle y,-x,\sqrt{4-x^2-y^2}\right\rangle\cdot\left\langle -\frac{x}{\sqrt{4-x^2-y^2}},\ -\frac{y}{\sqrt{4-x^2-y^2}},\ -1\right\rangle\\
&= -\sqrt{4-x^2-y^2}
\end{aligned}$$

and so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^2\int_0^{\sqrt{4-x^2}} -\sqrt{4-x^2-y^2}\,dy\,dx\\
&= -\int_0^2\left[\frac{y}{2}\sqrt{4-x^2-y^2}+\frac{4-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{4-x^2}}\right)\right]_0^{\sqrt{4-x^2}}\,dx\\
&= -\frac{\pi}{4}\int_0^2\left(4-x^2\right)\,dx = -\frac{\pi}{4}\cdot\frac{16}{3} = -\frac{4\pi}{3},
\end{aligned}$$

as before.

$\boxed{3}$ A simple parameterization of this surface is $\mathbf{r}(x,y) = \langle x, y, 4 - x^2 - y^2 \rangle$. Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle .$$

(Note that this is *not* properly oriented. We're told to use the "downward orientation" but here the $\mathbf{k}$ component is positive. Thus we should be using $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, -1 \rangle$.) Thus

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle x, y, 2z \rangle \cdot \langle -2x, -2y, -1 \rangle \; dx \; dy \\ &= \iint_D \left(-2x^2 - 2y^2 - 2z \right) \; dx \; dy. \end{aligned}$$

Now we notice that (according to our parameterization) $z = 4 - x^2 - y^2$. The region D in our parameter space is a unit square, so we get

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^1 \left[-2x^2 - 2y^2 - 2(4 - x^2 - y^2) \right) \; dx \; dy \\ &= \int_0^1 \int_0^1 (-8) \; dx \; dy = -8 \; \mathrm{Area}(D) = -8. \end{aligned}$$

$\boxed{4}$ One way to do this is to use the parameterization $\mathbf{r}(x,y) = \langle x, y, 4-x^2-y^2\rangle$, so $\mathbf{r}_x = \langle 1, 0, -2x\rangle$, $\mathbf{r}_y = \langle 0, 1, -2y\rangle$, and so

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle.$$

Thus we can write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \langle x, y, 2x+2y\rangle \cdot \langle 2x, 2y, 1\rangle = 2x^2 + 2y^2 + 2x + 2y.$$

Hence our flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(2x^2 + 2y^2 + 2x + 2y\right)\, dx\, dy,$$

where D is the unit disk. Thus we make the change to polar coordinates, where $2x^2+2y^2+2x+2y = 2r^2 + 2r\left(\cos(\theta)+\sin(\theta)\right)$ and $dx\, dy = r\, dr\, d\theta$. We get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^1 \left[2r^2 + 2r\left(\cos(\theta)+\sin(\theta)\right)\right] r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[2r^3 + 2r^2\left(\cos(\theta)+\sin(\theta)\right)\right] dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\left(\cos(\theta)+\sin(\theta)\right)\right] d\theta \\
&= \left[\frac{1}{2}\theta + \frac{2}{3}\left(\sin(\theta)-\cos(\theta)\right)\right]_0^{2\pi} = \pi.
\end{aligned}$$

Another approach is to use polar coordinates to parameterize the surface from the very beginning. That is, we could use the parameterization $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4-r^2\rangle$, in which case $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), -2r\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, so

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle.$$

In these coordinates our vector field is $\mathbf{F} = \langle r\cos(\theta), r\sin(\theta), 2r(\cos(\theta)+\sin(\theta))\rangle$, and therefore our flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^1 \langle r\cos(\theta), r\sin(\theta), 2r(\cos(\theta)+\sin(\theta))\rangle \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[2r^3 + 2r^2\left(\cos(\theta)+\sin(\theta)\right)\right] dr\, d\theta,
\end{aligned}$$

which is identical to an integral computed above.

$\boxed{5}$ Here we need to compute two integrals:

$$\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S},$$

where S_1 is the hemisphere (with outward-pointing normal) and S_2 is the unit disk in the xy-plane (with downward-pointing normal).

The integral over S_1 is very similar to the previous problem. We'll use the first parameterization of Problem 3, namely

$$\mathbf{r}(\phi, \theta) = \big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \big\rangle.$$

This gives us an outward-pointing normal

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin(\phi)\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \big\rangle,$$

so

$$\begin{aligned}
&\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \\
&= \big\langle -\cos(\phi), \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta) \big\rangle \cdot \big\langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \big\rangle \\
&= -\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^2(\phi)\cos(\phi)\sin(\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
&\iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} \\
&= \int_0^{\pi/2} \int_0^{2\pi} \big(-\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^2(\phi)\cos(\phi)\sin(\theta)\big)\, d\theta\, d\phi \\
&= \int_0^{\pi/2} \left(-\sin^2(\phi)\cos(\phi)\sin(\theta) - \sin^3(\phi) \cdot \frac{1}{2}\sin^2(\theta) - \sin^2(\phi)\cos(\phi)\cos(\theta)\right)\Bigg|_{\theta=0}^{2\pi} d\phi \\
&= 0.
\end{aligned}$$

The second integral is even simpler. Here S_2 is the unit disk in the xy-plane, with the unit normal $\mathbf{n} = -\mathbf{k}$ (straight down). Thus

$$\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{D} \big\langle -z, x, y \big\rangle \cdot \big\langle 0, 0, -1 \big\rangle\, dA = \iint\limits_{D} -y\, dA,$$

where D is the unit disk in the xy-plane. We use polar coordinates to compute this integral:

$$\begin{aligned}
\iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_{D} -y\, dA = \int_0^1 \int_0^{2\pi} -r\sin(\theta) \cdot r\, d\theta\, dr \\
&= \int_0^1 r^2\cos(\theta)\Big|_0^{2\pi} dr = 0.
\end{aligned}$$

Thus the full integral is

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint\limits_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 + 0 = 0.$$

We'll see a simpler way of computing this integral on Wednesday when we learn about the Divergence Theorem.