Stokes' Theorem

Cast of Players:

> S – an oriented, piecewise-smooth surface
>
> C – a simple, closed, piecewise-smooth curve that bounds S
>
> $\mathbf{F}$ – a vector field whose components have continuous derivatives
> in an open region of $\mathbf{R}^3$ containing S

Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

1. Suppose C is the curve obtained by intersecting the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clockwise when viewed from above. Let S be the inside of this ellipse, oriented with the upward-pointing normal. If $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. Suppose S is that part of the plane $x + y + z = 1$ in the first octant, oriented with the upward-pointing normal, and let C be its boundary, oriented counter-clockwise when viewed from above. If $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

3. Suppose S is a "light-bulb-shaped region" as follows. Imagine a light-bulb cut off at the base so that its boundary is the unit circle $x^2 + y^2 = 1$, oriented with the outward-pointing normal. (You can use either an old-fashioned light-bulb or a compact fluorescent if you're feeling green.) Suppose $\mathbf{F} = \left\langle e^{z^2 - 2z}x, \sin(xyz) + y + 1, e^{z^2}\sin(z^2) \right\rangle$. Compute the flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.
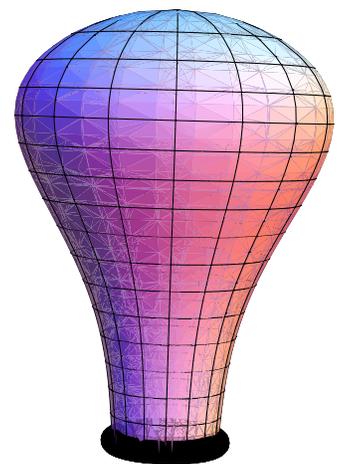
4 Suppose $\mathbf{F} = \langle -y, x, z \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 4$, oriented with the outward-pointing normal (so that the normal at $(5, 0, 0)$ is $\mathbf{i}$). Compute the flux integral $\iint\limits_{S} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

5 Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. What can you say about $\iint\limits_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ and $\iint\limits_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$?

6 Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. Is it true that $\iint\limits_{S_1} \mathbf{G} \cdot d\mathbf{S} = \iint\limits_{S_2} \mathbf{G} \cdot d\mathbf{S}$ for any vector field $\mathbf{G}$? That is, can you *always* choose the easiest surface to work with when computing flux integrals over a surface with boundary?

7 Suppose S is a closed surface (that is, a surface without a boundary). Must $\iint\limits_{S} \mathbf{F} \cdot d\mathbf{S} = 0$?

Stokes' Theorem – Answers and Solutions

$\boxed{1}$ There are two integrals to compute here, so we do them both.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ The ellipse is a graph (using $z = x$) over the unit circle in the xy-plane. Thus we can parameterize it as $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t) \rangle$ for $0 \leq t \leq 2\pi$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos(t), \cos(t), 2\sin(t) \rangle$$
$$d\mathbf{r} = \langle -\sin(t), \cos(t), -\sin(t) \rangle \, dt$$

and so

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \left(-\sin(t)\cos(t) + \cos^2(t) - 2\sin^2(t) \right) \, dt$$
$$= \left(-\sin(t)\cos(t) + 1 - 3\sin^2(t) \right) \, dt.$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\sin(t)\cos(t) + 1 - 3\sin^2(t) \right) \, dt$$
$$= \left(-\frac{1}{2}\sin^2(t) + t - \frac{3}{2}t + \frac{3}{4}\sin(2t) \right) \Big|_0^{2\pi}$$
$$= -\pi.$$

The flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ Again the elliptical disk is a graph (using $z = x$) over the unit disk in the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, x \rangle$ for $x^2 + y^2 \leq 1$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get $\operatorname{curl} \mathbf{F} = \langle 1, 0, 0 \rangle$ and $\mathbf{r}_x \times \mathbf{r}_y = \langle -1, 0, 1 \rangle$. Thus

$$\iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{\text{unit disk}} \langle 1, 0, 0 \rangle \cdot \langle -1, 0, 1 \rangle \, dx \, dy$$
$$= -\iint_{\text{unit disk}} 1 \, dx \, dy$$
$$= -\pi,$$

since the last integral is simply the area $\pi(1)^2 = \pi$ of the unit disk.

Note that the two integrals agree. Another victory for Stokes' theorem!

There are two integrals to compute here, so we do them both. Here's a picture of the surface and curve, so we're all on the same page:
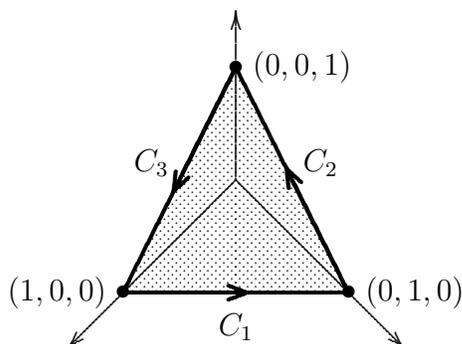
The region S is the dotted triangle (with the upward normal coming straight toward the viewer) and the curve C is the union $C_1 \cup C_2 \cup C_3$.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ This integral will really be the sum of three separate integrals, over each of C_1, C_2, and C_3. We begin with C_1. A simple parameterization of this line segment is

$$
\begin{aligned}
\mathbf{r}(t) &= \text{starting point} + t \, (\text{ ending point } - \text{ starting point }) \\
&= \langle 1, 0, 0 \rangle + t \, (\langle 0, 1, 0 \rangle - \langle 0, 1, 0 \rangle) \\
&= \langle 1 - t, t, 0 \rangle.
\end{aligned}
$$

Thus $d\mathbf{r} = \langle -1, 1, 0 \rangle \, dt$. In terms of this parameterization, the vector field $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$ becomes

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) &= \langle (1 - t)^2 - t^2, t^2 - 0^2, 0^2 - (1 - t)^2 \rangle \\
&= \langle 1 - 2t, t^2, -(1 - t)^2 \rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \langle 1 - 2t, t^2, -(1 - t)^2 \rangle \cdot \langle -1, 1, 0 \rangle \ dt \\
&= \left(t^2 + 2t - 1 \right) \ dt.
\end{aligned}
$$

Thus

$$
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2 + 2t - 1 \right) \ dt = \frac{1}{3}.
$$

The curve C_2 is similar: it's parameterized by $\mathbf{r}(t) = \langle 0, 1 - t, t \rangle$, so

$$
\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \langle -(1 - t)^2, 1 - 2t, t^2 \rangle \cdot \langle 0, -1, 1 \rangle \ dt \\
&= \left(t^2 + 2t - 1 \right) \ dt.
\end{aligned}
$$

Thus

$$
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2 + 2t - 1 \right) \ dt = \frac{1}{3},
$$

and very similarly $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}$ as well. Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

The flux integral $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ This triangular surface is a graph (using $z = 1 - x - y$) over the triangle $T = \{(x, y) \; : \; 0 \le y \le 1 - x, \; 0 \le x \le 1\}$ in the first quadrant of the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle$ for $(x, y) \in T$. Since $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, we get

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & y^2 - z^2 & z^2 - x^2 \end{vmatrix} = \langle 2z, 2x, 2y \rangle.$$

A similar computation shows that and $\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 1, 1 \rangle$ (and, since the $\mathbf{k}$ coefficient is positive, this is the upward-pointing normal). Thus

$$\iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_T \langle 2(1 - x - y), 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dA$$
$$= \int_0^1 \int_0^{1-x} 2 \, dy \, dx$$
$$= 1,$$

as before. More success for Stokes' theorem!

$\boxed{3}$ The point of this problem is to find use Stokes' theorem to avoid computing the flux integral over S (whatever confusing surface that could be) and instead compute the line integral over the unit circle C in the xy-plane. We use the parameterization $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$, so

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos(t), \sin(t) + 1, 0 \rangle$$
$$d\mathbf{r} = \langle -\sin(t), \cos(t), 0 \rangle \, dt$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \cos(t) \, dt.$$

Thus

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \cos(t) \, dt = 0.$$

That was pretty easy.

$\boxed{4}$ Again we integrate the line integral over the boundary curve C rather than the flux integral over the (more complicated) surface S. The boundary curve is the circle $x^2 + y^2 + 4^2 = 25$ (or $x^2 + y^2 = 9$) in the plane $z = 4$, but a note of caution is in order. The natural parameterization

(or the one we usually think of) is $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4\rangle$ actually parameterizes $-C$ (that is, C with the opposite orientation)! Why is that? Imagine a person walking this boundary with their head in the normal (outward) direction. The remaining part of the sphere is on their *right* if they're walking counter-clockwise. It should be on their left, so they should be walking clockwise.

We'll calculate $\oint_{-C} \mathbf{F} \cdot d\mathbf{r}$ anyway, since we like the parameterization. In terms of this parameterization,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -3\sin(t), 3\cos(t), 4\rangle$$
$$d\mathbf{r}(t) = \langle -3\sin(t), 3\cos(t), 0\rangle \, dt$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}(t) = 9 \, dt.$$

Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 9 \, dt = 18\pi,$$

and so $\displaystyle \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -18\pi$.

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5 If the boundaries of S_1 and S_2 are both C (with the same orientation!), then two applications of Stokes' theorem means that

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

so these two flux integrals must be the same.

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6 What the previous problem was getting at was that this is always true when $\mathbf{G} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$. But, alas, it is not true in general.

Here's a "simple" example: Let $\mathbf{G} = \langle 0, 0, z\rangle$ and S_1 be the unit square in the xy-plane:

$$S_1 = \{(x, y, z) \ : \ 0 \le x \le 1, \ 0 \le y \le 1, \ z = 0\}.$$

Then we'll let S_2 be the rest of the boundary of the unit cube, oriented so $S_1 \cup S_2$ encloses this cube and we have the outward-pointing normal. It should be clear that the flux across S_1 is zero (since $\mathbf{G} = \mathbf{0}$ on this surface), but a computation shows that, in fact, $\iint_{S_2} \mathbf{G} \cdot d\mathbf{S} = 1$.

We'll see in the next section that what is claimed here is true when $\operatorname{div} \mathbf{G} = 0$. But, like $\operatorname{curl} \mathbf{F} = \mathbf{0}$ implying that $\mathbf{F} = \nabla f$, it turns out that (under suitable assumptions) $\operatorname{div} \mathbf{G} = 0$ implies $\mathbf{G} = \operatorname{curl} \mathbf{F}$. (This will also give us an easy way to compute our "simple" example.)

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7 No, $\displaystyle \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ in this case, but not the given integral $\displaystyle \iint_S \mathbf{F} \cdot d\mathbf{S}$.

If you're wondering how, say, the total net flow in or out of a closed surface can be something other than zero, then chances are you're too focused on water. A liquid like water is *incompressible*, so for water that flows into a region must be balanced by an equal amount that flows out (assuming that this region is totally submerged / full of water). But this is not true of all things that can flow; for example, electrical charge or temperature (two examples we had in homework due today) can have "sinks" or "sources" as we'll see Friday.