Integral Theorems Review

1 For these problems, find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ and C are as given.

(a) $\mathbf{F} = \langle x, y, z \rangle$ and C is parameterized by $\mathbf{r}(t) = \langle t, t, t \rangle$ $(0 \le t \le 1)$

(b) $\mathbf{F} = \langle x, y, z \rangle$ and C is parameterized by $\mathbf{r}(t) = \langle t, \sqrt{t}, \sqrt[3]{t} \rangle$ $(0 \le t \le 1)$

(c) $\mathbf{F} = \langle 2xy, x^2 + z, y + 2z \rangle$ and C is parameterized by

$$\mathbf{r}(t) = \langle t^2 - t,\ \sin(\pi t),\ \cos^2(\pi t) \rangle \qquad (0 \le t \le 1)$$

(d) $\mathbf{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the unit circle in the xy-plane, oriented counter-clockwise.

2 For these problems, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ and S are as given.

(a) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and S is the hemisphere of radius 1, centered at the origin and above the xy-plane, oriented with the upward-pointing normal.

(b) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and S is the solid disk of radius 1 in the xy-plane, centered at the origin, oriented with the upward-pointing normal.

(c) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and $S = S_{(a)} \cup \left(-S_{(b)}\right)$ is the union of the two surfaces from parts (a) and (b), oriented with the outward-pointing normals.

(d) $\mathbf{F} = \langle x^2 y - 3x,\ -xy^2 + 2\cos(y)z,\ \sin(y)z^2 \rangle$ and $S = S_{(a)} \cup \left(-S_{(b)}\right)$ is the same as in part (c).

3 For these problems, find $\iiint_E \operatorname{div} \mathbf{F}\, dV$, where $\mathbf{F}$ and E are as given.

(a) $\mathbf{F} = \operatorname{curl} \mathbf{G}$ (where $\mathbf{G}$ is any appropriately smooth vector field) and E is any simple solid

(b) $\mathbf{F} = \langle x, y^2, -2yz \rangle$ and E is the solid ball of radius a centered at the origin

(c) $\mathbf{F} = \langle x^2, 2yz, x^2 - z^2 \rangle$ and E is the unit cube with corners at $(0, 0, 0)$ and $(1, 1, 1)$

Integral Theorems Review – Answers and Solutions

1 (a) To compute this we "just do it". Using $\mathbf{r}(t) = \langle t, t, t\rangle$, we get $\mathbf{F}(\mathbf{r}(t)) = \langle t, t, t\rangle$ and $d\mathbf{r} = \langle 1, 1, 1\rangle\, dt$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t,t,t\rangle \cdot \langle 1,1,1\rangle\, dt = \int_0^1 3t\, dt = \frac{3}{2}$$

(b) Here we could "just do it" again, but the parameterization is messier. It's simpler to notice that the vector field $\mathbf{F} = \langle x, y, z\rangle$ is conservative and thus independent of path. We see this by noticing that

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

or simply that $\mathbf{F} = \nabla f$ where $f = \frac{1}{2}(x^2 + y^2 + z^2)$. Since the path in part (b) starts and ends at the same places as the path in part (a), we can simply integrate over that path. This is what we've done in part (a), so we get the same answer: $\frac{3}{2}$.
Another approach would be to use the fundamental theorem of line integrals:

$$\begin{aligned}\int_C \nabla f\cdot d\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,1,1) - f(0,0,0) \\ &= \frac{1}{2}\left(1^2 + 1^2 + 1^2\right) - 0 = \frac{3}{2},\end{aligned}$$

as before.

(c) Here the key is that the given vector field is conservative. We can see this by computing $\operatorname{curl}\mathbf{F} = 0$ or writing $\mathbf{F} = \nabla f$, where $f = x^2y + yz + z^2$. Since C is a closed curve (one that starts and ends at the point $(0,0,1)$), the integral must be zero. We could also see this via the fundamental theorem of line integrals:

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(0,0,1) - f(0,0,1) = 0.$$

It's also possible to see this using Stokes' theorem (where S is some (any!) oriented surface with C as its boundary):

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

Either approach is fine.

(d) This is a cautionary tale. Let's think of our vector as $\mathbf{F} = \langle P, Q\rangle$. Then it's straightforward to see that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

so we might think that we can proceed as in part (c). That is, we could say that $\mathbf{F}$ is conservative and thus $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. Or we might apply Green's theorem: let D be the unit disk (with C as it's boundary), so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\, dA = 0.$$

Both these conclusions are **WRONG!** and in fact the line integral has value 2π.

What has gone wrong? We've tried to blindly apply theorems without checking the hypotheses. In the first case we tried to apply theorem 6 (on page 928) that says that $\mathbf{F}$ is conservative when $P_y = Q_x$. But we must have this equality on a simply connected region D. In our case, both P and Q are undefined at the origin $(0, 0)$, so any region D containing the unit circle also contains a hole at the origin. Similarly, Green's theorem require that P and Q have continuous derivatives inside D, which again fails at the origin.

We can compute the actual value using a simple parameterization of C:

$$\mathbf{r}(t) = \langle x, y\rangle = \langle \cos(t), \sin(t)\rangle \qquad \text{and so} \qquad d\mathbf{r}(t) = \langle -\sin(t), \cos(t)\rangle \; dt.$$

Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle -\frac{\sin(t)}{1}, \frac{\cos(t)}{1} \right\rangle \cdot \langle -\sin(t), \cos(t)\rangle \; dt \\
&= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t)\right) \; dt \\
&= \int_0^{2\pi} 1 \; dt = 2\pi,
\end{aligned}$$

as claimed.

2 (a) While we could compute this directly, it seems easier to use Stokes' theorem to compute a line integral instead. Here the boundary is simply the unit circle in the xy-plane, so we can parameterize C as

$$\mathbf{r}(t) = \langle x, y, z\rangle = \langle \cos(t), \sin(t), 0\rangle.$$

Thus the vector we are integrating is

$$\langle 2y, z - 2x, yz\rangle = \langle 2\sin(t), -2\cos(t), 0\rangle$$

while $d\mathbf{r} = \langle -\sin(t), \cos(t), 0\rangle \; dt$. Thus

$$\begin{aligned}
\iint_S \operatorname{curl}\langle 2y, z - 2x, yz\rangle \cdot d\mathbf{S} &= \oint_C \langle 2y, z - 2x, yz\rangle \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \langle 2\sin(t), -2\cos(t), 0\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle \; dt \\
&= \int_0^{2\pi} \left(-2\sin^2(t) - 2\cos^2(t)\right) \; dt \\
&= \int_0^{2\pi} -2 \; dt = -2t\Big|_0^{2\pi} = -4\pi.
\end{aligned}$$

(b) As with part (a), we could compute this integral directly. Now, however, we have even more of an incentive to use Stokes' theorem and compute the line integral – we've already computed this line integral! That is, our surface has the same boundary C (the unit circle in the xy-plane) as the surface from part (a), so we can simply use our answer from there:

$$\iint_S \operatorname{curl}\langle 2y, z - 2x, yz\rangle \cdot d\mathbf{S} = \oint_C \langle 2y, z - 2x, yz\rangle \cdot d\mathbf{r} = -4\pi.$$

(c) Our surface is $S_{(a)} \cup S_{(-b)}$, the union of the surface from part (a) (with the same outward-pointing normal) and the surface from part (b) (with the downward-pointing normal, the opposite orientation from part (b)). Thus

$$\begin{aligned}
\iint_S \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} &= \iint_{S_{(a)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} + \iint_{-S_{(b)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} \\
&= \iint_{S_{(a)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} - \iint_{S_{(b)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} \\
&= (-4\pi) - (-4\pi) = 0.
\end{aligned}$$

Notice that it doesn't matter what the values of the surface integrals from part (a) and part (b) were, it only matters that they were the same. Thus the given surface integral over the closed surface $S_{(a)} \cup -S_{(b)}$ will always be zero. This is because of two facts:

(i) Both $S_{(a)}$ and $S_{(b)}$ are oriented surfaces that have the curve C (as oriented) as their boundary, and

(ii) The flux integrals over $S_{(a)}$ and $S_{(b)}$ involve integrating a vector field that is a curl, and thus we can apply Stokes' theorem.

Another approach is to apply the Divergence theorem. That is, since we're integrating over a closed surface S that bounds a solid hemisphere (we'll call this E), then

$$\iint_S \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} = \iiint_E \operatorname{div}\left(\operatorname{curl}\langle 2y, z-2x, yz\rangle\right)\, dV.$$

Now the key is the $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ for any vector field $\mathbf{G}$, so this integral is zero.

(d) This is very similar to part (c). Perhaps the simplest approach from (c) that we could use here is the last one, using the Divergence theorem. In this approach we again let E be the solid unit hemisphere that has S as its (oriented) boundary. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\operatorname{div}\mathbf{F})\, dV,$$

where $\mathbf{F} = \langle x^2y - 3x,\ -xy^2 + 2\cos(y)z,\ \sin(y)z^2\rangle$ so

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}\left(x^2y - 3x\right) + \frac{\partial}{\partial y}\left(-xy^2 + 2\cos(y)z\right) + \frac{\partial}{\partial z}\left(\sin(y)z^2\right) \\
&= 2xy + (-2xy - 2\sin(y)z) + 2\sin(y)z = -3.
\end{aligned}$$

Thus our flux integral is (again!) zero.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (-3)\, dV = -3\mathrm{Vol}(E) = -3\left(\frac{2}{3}\pi\right) = -2\pi,$$

where we've used the fact that the volume of a hemisphere is $\frac{2}{3}\pi r^3$ (and here $r = 1$).

3 (a) The key here is that $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ (this is just an application of Clairaut's theorem on order of partial derivatives). Thus

$$\iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{G})\ dV = \iiint_E 0\ dV = 0.$$

(b) Here

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1.$$

Thus

$$\iiint_E \operatorname{div}\mathbf{F}\ dV = \iiint_E 1\ dV = \operatorname{Vol}(E) = \frac{4}{3}\pi a^3,$$

since E is a ball of radius a.

(c) Here

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(x^2 - z^2) = 2x + 2z - 2z = 2x.$$

Thus, since $E = \{(x, y, z) \ :\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$, we get

$$\iiint_E \operatorname{div}\mathbf{F}\ dV = \iiint_E 2x\ dV = \int_0^1 \int_0^1 \int_0^1 2x\ dx\ dy\ dz = 1.$$