Math 21a Review Session for Exam 2

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1. Suppose that the temperature distribution in a room is given by $T = T(x, y, z)$, and that $T(1, 3, 2) = 55$, $T_x(1, 3, 2) = -1$, $T_y(1, 3, 2) = 2$, and $T_z(1, 3, 2) = 3$.

   (a) At $(1, 3, 2)$, in which direction is the temperature increasing most rapidly, and at what rate is it increasing?

   (b) Write an equation for the tangent plane to the isotherm (level surface) $T(x, y, z) = 55$ at the point $(1, 3, 2)$.

   (c) Use a linear approximation to estimate $T(1.01, 2.98, 2.10)$.

   (d) A fly moves through the room according to $\vec{r}(t) = \langle 2 + \cos \pi t, 3 + 2 \sin \pi t, 2t \rangle$. How fast is the temperature the fly experiences changing when it passes through the point $(1, 3, 2)$?

2. Show that the sphere $x^2 + y^2 + z^2 = r^2$ and the elliptical cone $z^2 = a^2x^2 + b^2y^2$ are orthogonal at every point of intersection. Why doesn't this depend on a and b?

3. Let $f(x, y) = x^2 + y^2 - a(x - 1)^2 - b(y - 1)^2$.

   (a) Show that if neither a nor b is equal to 1, f has exactly one critical point.

   (b) Again assuming $a, b \neq 1$, give conditions on a and b under which the critical point you found in part (a) is, in turn, a local maximum, a local minimum, and a saddle.

   (c) What happens when $a = b = 1$?

4. (11.7, # 12, modified)

   (a) Find and classify all critical points of $f(x, y) = x^2 + y^2 + \frac{1}{x^2y^2}$.

   (b) Find the maximum and minimum values of f (if they exist) along the curve $xy = 4$.

5. (a) True or false? $g(x, y) = e^{f(x,y)}$ has a maximum/minimum/saddle point at (a, b) if and only if $f(x, y)$ does.

   (b) More generally, how do the maxima/minima/saddle points of $h(x, y) = g(f(x, y))$ depend on the properties of g and f?

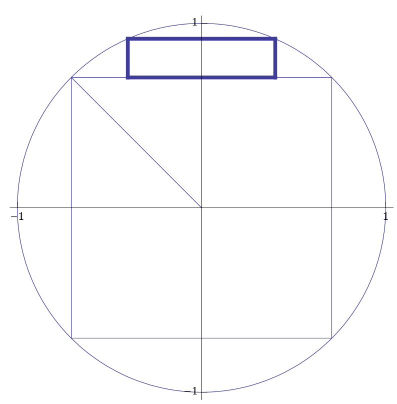
6. (11.7, # 31, modified) Show that $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$ has exactly two critical points, and that both are local maxima.

7. (a) Find where the tangent plane to $z^2 - x^2 - y^2 = 1$ is parallel to the plane $3x - y + z = 4$. If no such point exists, explain why not.

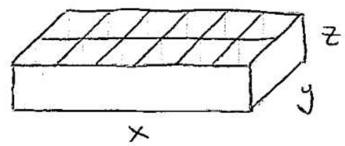
   (b) What is the "steepest" a tangent plane to $z^2 - x^2 - y^2 = 1$ can be? In other words, what is the largest possible angle between the z-axis and the normal vector of a tangent plane to $x^2+y^2-z^2 = 1$?

8. (a) Show that the maximum cross-sectional area of a rectangular beam cut from a circular log of radius 1 foot occurs when the cross-section is a square.

   (b) After cutting the beam in part (a), four pieces of wood are left over. If each piece to be then trimmed to form a rectangular beam, as shown in bold in the diagram below, find the dimensions that maximize the total cross-sectional area.



9. An ice cube tray is designed with 12 compartments, as shown in the diagram below. Find the dimensions that minimize the cost of the tray, subject to the constraints that each compartment has a square horizontal cross-section, and that the total volume (ignoring the partitions) is 12 cubic inches.



10. Compute the iterated integral

$$\int_0^1 \int_{\tan^{-1} x}^{\pi/4} x dy dx.$$

11. Let D_1 be the region bounded by the polar curve $r(\theta) = R + \cos\theta$, where $R \geq 1$.

    (a) Compute the area of D_1.

    (b) Explain why D_1 is *not* a circle.

    (c) Let D_2 be the circle $x^2 + y^2 = R^2$. Show that the ratio of the areas of D_1 and D_2 approaches 1 as R goes to infinity.

12. Let S be the solid region in the first octant bounded by the surface $z = xy^2$, the cylinder $x^2 + y^2 = 4$, and the xy-plane.

    (a) What is the highest point in the region S?

    (b) Sketch S.

    (c) Compute the volume of S.

13. Compute the volume of the solid in the first octant bounded by the planes $2x + y + 3z = 6, 2x - y = 0, x = 0$, and $z = 0$.