Math 21a Review Session for Exam 2
Solutions to Selected Problems

John Hall

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Note: Problems which do not have solutions were done in the review session.

1. Suppose that the temperature distribution in a room is given by $T = T(x, y, z)$, and that $T(1, 3, 2) = 55, T_x(1, 3, 2) = -1, T_y(1, 3, 2) = 2$, and $T_z(1, 3, 2) = 3$.

   (a) At $(1, 3, 2)$, in which direction is the temperature increasing most rapidly, and at what rate is it increasing?

   (b) Write an equation for the tangent plane to the isotherm (level surface) $T(x, y, z) = 55$ at the point $(1, 3, 2)$.

   (c) Use a linear approximation to estimate $T(1.01, 2.98, 2.10)$.

   (d) A fly moves through the room according to $\vec{r}(t) = \langle 2 + \cos \pi t, 3 + 2 \sin \pi t, 2t\rangle$. How fast is the temperature the fly experiences changing when it passes through the point $(1, 3, 2)$?

2. Show that the sphere $x^2 + y^2 + z^2 = r^2$ and the elliptical cone $z^2 = a^2x^2 + b^2y^2$ are orthogonal at every point of intersection. Why doesn't this depend on a and b?

3. Let $f(x, y) = x^2 + y^2 - a(x - 1)^2 - b(y - 1)^2$.

   (a) Show that if neither a nor b is equal to 1, f has exactly one critical point.

   (b) Again assuming $a, b \neq 1$, give conditions on a and b under which the critical point you found in part (a) is, in turn, a local maximum, a local minimum, and a saddle.

(c) What happens when $a = b = 1$?

4. (11.7, # 12, modified)

(a) Find and classify all critical points of $f(x, y) = x^2 + y^2 + \frac{1}{x^2y^2}$.

Solution: Set $\nabla f = \langle 2x - \frac{2}{x^3y^2}, 2y - \frac{2}{x^2y^3}\rangle = \langle 0, 0\rangle$ to get $y^2 = \frac{1}{x^4}$ and $x^2 = \frac{1}{y^4}$. Combining these gives $y^4 = \frac{1}{x^8} = \frac{1}{x^2}$, i.e., $x^6 = 1$, i.e., $x = \pm 1$, and so $y = \pm 1$. So the critical points are $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. To classify these critical points, we use the Second Derivative Test:

$$D(x, y) = \begin{vmatrix} 2 + \frac{6}{x^4y^2} & \frac{4}{x^3y^3} \\ \frac{4}{x^3y^3} & 2 + \frac{6}{x^2y^4} \end{vmatrix}$$

$$\begin{aligned} D(1, 1) &= D(-1, -1) = \begin{vmatrix} 8 & 4 \\ 4 & 8 \end{vmatrix} = 64 - 16 = 48 \\ D(1, -1) &= D(1, -1) = \begin{vmatrix} 8 & -4 \\ -4 & 8 \end{vmatrix} = 64 - 16 = 48 \end{aligned}$$

Since in all cases $D > 0$ and $f_{xx} > 0$, each of the four critical points is a local minimum.

(b) Find the maximum and minimum values of f (if they exist) along the curve $xy = 4$.

Solution: Using the method of Lagrange Multipliers, we have

$$\begin{aligned} 2x - \frac{2}{x^3y^2} &= \lambda y \\ 2y - \frac{2}{x^2y^3} &= \lambda x \\ xy &= 4 \end{aligned}$$

Multiplying the first equation by x and the second by y gives

$$\begin{aligned} 2x^2 - \frac{2}{x^2y^2} &= \lambda xy \\ 2y^2 - \frac{2}{x^2y^2} &= \lambda xy \end{aligned}$$

Then, since $xy = 4$, we have

$$\begin{aligned} 2x^2 - \frac{2}{16} &= 4\lambda \\ 2y^2 - \frac{2}{16} &= 4\lambda \end{aligned}$$

Equating these two expressions gives $y = \pm x$. Using the constraint $xy = 4$ gives $(2, 2)$ and $(-2, -2)$. $f(2, 2) = f(-2, -2) = 8 + \frac{1}{16}$, and this is clearly the absolute minimum, since $f(x, y) \to \infty$ as x or y goes off to ∞.

5. (a) True or false? $g(x, y) = e^{f(x,y)}$ has a maximum/minimum/saddle point at (a, b) if and only if $f(x, y)$ does.

   (b) More generally, how do the maxima/minima/saddle points of $h(x, y) = g(f(x, y))$ depend on the properties of g and f?

   Solution: By the Chain Rule,

$$\nabla h(x, y) = \langle g'(f(x, y))f_x(x, y), g'(f(x, y))f_y(x, y)\rangle = g'(f(x, y))\nabla f(x, y).$$

So $\nabla h(a, b) = \langle 0, 0\rangle$ if $g'(f(a, b)) = 0$ or $\nabla f(a, b) = \langle 0, 0\rangle$. We'll look at these two cases separately, but first we compute

$$\begin{aligned} D &= \begin{vmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{vmatrix} \\ &= \begin{vmatrix} g''(f)f_x^2 + g'(f)f_{xx} & g''(f)f_xf_y + g'(f)f_{xy} \\ g''(f)f_yf_x + g'(f)f_{yx} & g''(f)f_y^2 + g'(f)f_{yy} \end{vmatrix} \end{aligned}$$

Case 1: $g'(f(a, b)) = 0$
In this case,

$$\begin{aligned} D(a, b) &= \begin{vmatrix} g''(f(a, b))f_x^2(a, b) & g''(f(a, b))f_x(a, b)f_y(a, b) \\ g''(f(a, b))f_y(a, b)f_x(a, b) & g''(f(a, b))f_y^2(a, b) \end{vmatrix} \\ &= (g''(f(a, b)))^2 \begin{vmatrix} f_x^2(a, b) & f_x(a, b)f_y(a, b) \\ f_y(a, b)f_x(a, b) & f_y^2(a, b) \end{vmatrix} \\ &= (g''(f(a, b)))^2 \left(f_x^2(a, b)f_y^2(a, b) - f_x^2(a, b)f_y^2(a, b)\right) \\ &= 0. \end{aligned}$$

So the Second Deriavtive Test tells us nothing. However, we can still say something the maxima and minima of h. Specifically, if

$g(u)$ has a local maximum at $u = f(a,b)$, then $h(x,y)$ has a local maximum at (a,b). To see this, note that $g(u) \leq g(f(a,b))$ for all u near $f(a,b)$. Assuming $f(x,y)$ is continuous (in this class functions are generally assumed to be continuous unless stated otherwise), this means that $g(f(x,y)) \leq g(f(a,b))$ for all (x,y) near (a,b). By similar reasoning, if $g(u)$ has a local minimum at $u = f(a,b)$, then $h(x,y)$ has a local minimum at (a,b).
Case 2: $\nabla f(a,b) = \langle 0,0 \rangle$
In this case,

$$\begin{aligned} D(a,b) &= \begin{vmatrix} g'(f(a,b))f_{xx}(a,b) & g'(f(a,b))f_{xy}(a,b) \\ g'(f(a,b))f_{yx}(a,b) & g'(f(a,b))f_{yy}(a,b) \end{vmatrix} \\ &= (g'(f(a,b)))^2 \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} \end{aligned}$$

So, assuming $g'(f(a,b)) \neq 0$, $D(a,b)$ has the same sign as

$$\begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}.$$

So in the case $\nabla f(a,b) = \langle 0,0 \rangle$ and $g'(f(a,b)) \neq 0$, $h(x,y)$ has a saddle at (a,b) if and only if $f(x,y)$ does. However, the sign of $h_{xx}(a,b) = g'(f(a,b))f_{xx}(a,b)$ depends on both $g'(f(a,b))$ and $f_{xx}(a,b)$. If $f(x,y)$ has a local minimum at (a,b) and $g'(f(a,b)) > 0$, then $h(x,y)$ has a local minimum at (a,b). Similarly, if $f(x,y)$ has a local maximum at (a,b) and $g'(f(a,b)) > 0$, then $h(x,y)$ has a local maximum at (a,b). If $g'(f(a,b)) < 0$, these conclusions are reversed.

6. (11.7, # 31, modified) Show that $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$ has exactly two critical points, and that both are local maxima.

   Solution: Setting

$$\begin{aligned} \nabla f &= \langle -4x(x^2-1) - 2(2xy-1)(x^2y-x-1), -4x(x^2y-x-1) \rangle \\ &= \langle 0,0 \rangle \end{aligned}$$

   gives

$$\begin{aligned} -4x(x^2-1) - 2(2xy-1)(x^2y-x-1) &= 0 \\ -2x^2(x^2y-x-1) &= 0. \end{aligned}$$

The second equation gives $x = 0$ or $x^2y - x - 1 = 0$. If $x = 0$, then the first equation becomes $-2 = 0$, which has no solutions. If $x^2y-x-1 = 0$, then the first equation becomes $-4x(x^2-1) = 0$, which implies $x = 0, 1$, or -1. If $x = 0$, then the equation $x^2y - x - 1 = 0$ becomes $-1 = 0$, which again has no solutions. If $x = 1$, we get $y-2 = 0$, i.e., $y = 2$. If $x = -1$, we get $y = 0$. So there are two critical points, $(1, 2)$ and $(-1, 0)$.

To classify these critical points, we use the second derivative test.

$$D = \begin{vmatrix} -4(x^2-1) - 8x^2 - 4y(x^2y-x-1) - 2(2xy-1)^2 & -4x(x^2y-x-1) - 2x^2(2xy-1) \\ -4x(x^2y-x-1) - 2x^2(2xy-1) & -2x^4 \end{vmatrix}$$

$$\begin{aligned} D(1,2) &= \begin{vmatrix} -26 & -6 \\ -6 & -2 \end{vmatrix} = 52 - 36 = 16 \\ D(-1,0) &= \begin{vmatrix} -10 & 2 \\ 2 & -2 \end{vmatrix} = 20 - 4 = 16 \end{aligned}$$

Each critical point is a local maximum, since $D > 0$ and $f_{xx} < 0$.

7. (a) Find where the tangent plane to $z^2 - x^2 - y^2 = 1$ is parallel to the plane $3x - y + z = 4$. If no such point exists, explain why not.

   Solution: Letting $f(x, y, z) = z^2 - x^2 - y^2$, we need to find where $\nabla f = \lambda\langle 3, -1, 1\rangle$. Since $\nabla f = \langle -2x, -2y, 2z\rangle$, this gives

$$\begin{aligned} -2x &= 3\lambda \\ -2y &= -\lambda \\ 2z &= \lambda \\ z^2 - x^2 - y^2 &= 1 \end{aligned}$$

   Solving the first three equations for x, y, and z, respectively, gives $x = -\frac{3}{2}\lambda, y = \frac{1}{2}\lambda$, and $z = \frac{1}{2}\lambda$. Plugging these expressions into the last equation gives $\frac{1}{4}\lambda^2 - \frac{9}{4}\lambda^2 - \frac{1}{4}\lambda^2 = 1$, i.e., $-\frac{9}{4}\lambda^2 = 1$. This last equation has no real solutions, so there is no point on the surface $z^2 - x^2 - y^2 = 1$ where the tangent plane is parallel to the plane $3x - y + z = 4$.

   (b) What is the "steepest" a tangent plane to $z^2 - x^2 - y^2 = 1$ can be? In other words, what is the largest possible angle between the z-axis and the normal vector of a tangent plane to $x^2+y^2-z^2 = 1$?

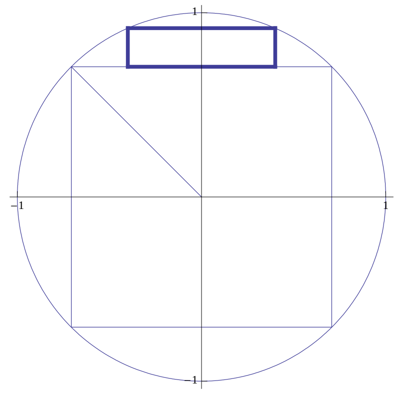
Solution: Note that the surface is a hyperboloid of two sheets, opening along the z-axis. Let θ be the angle between the gradient vector $\nabla f = \langle -2x, -2y, 2z\rangle$ and $\vec{k} = \langle 0, 0, 1\rangle$. Then

$$\begin{aligned}
\cos\theta &= \frac{\nabla f \cdot \vec{k}}{|\nabla f||\vec{k}|} \\
&= \frac{\langle -2x, -2y, 2z\rangle \cdot \langle 0, 0, 1\rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\
&= \frac{2z}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\
&= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
&= \frac{z}{\sqrt{2z^2 - 1}},
\end{aligned}$$

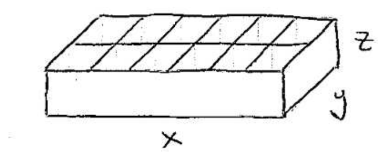
since $x^2 + y^2 = z^2 - 1$ on the surface. Assuming $z > 0$ (the two halves of the hyperboloid are symmetric), the angle θ will be largest where $\cos\theta$ is smallest. So we need to minimize the function $g(z) = \frac{z}{\sqrt{2z^2-1}}$ on the domain $z \geq 1$. Since $g'(z) = \frac{\sqrt{2z^2-1} - 2z^2/\sqrt{2z^2-1}}{(2z^2-1)} = \frac{-1}{(2z^2-1)^{3/2}} < 0$ for all z, there is no minumim value of $\cos\theta$; in fact $\cos\theta$ approaches $\frac{\sqrt{2}}{2}$ as $z \to \infty$. This means that the angle between the z-axis and the normal vector of a tangent plane to $x^2 + y^2 - z^2 = 1$ is always less than $\frac{\pi}{4}$, and approaches $\frac{\pi}{4}$ as we move further and further up the surface.

8. (a) Show that the maximum cross-sectional area of a rectangular beam cut from a circular log of radius 1 foot occurs when the cross-section is a square.

   (b) After cutting the beam in part (a), four pieces of wood are left over. If each piece to be then trimmed to form a rectangular beam, as shown in bold in the diagram below, find the dimensions that maximize the total cross-sectional area.



9. An ice cube tray is designed with 12 compartments, as shown in the diagram below. Find the dimensions that minimize the cost of the tray, subject to the constraints that each compartment has a square horizontal cross-section, and that the total volume (ignoring the partitions) is 12 cubic inches.



Solution: We want to minimize the cost function $C(x, y, z) = xy + 7xz + 3yz$ subject to the constraints $xyz = 12$ and $\frac{x}{6} = \frac{y}{2}$, which we can write as $x - 3y = 0$. Using Lagrange Multipliers we get the following system of five equations and five unknowns:

$$\begin{aligned} y + 7z &= \lambda yz + \mu \\ x + 3z &= \lambda xz - 3\mu \\ 7x + 3y &= \lambda xy \\ xyz &= 12 \\ x - 3y &= 0 \end{aligned}$$

Adding 3 times the first equation to the second gives $x + 3y + 24z = 3\lambda yz + \lambda xz$, and replacing x by $3y$ in this and the remaining equations gives

$$\begin{aligned} 6y + 24z &= 6\lambda yz \\ 24y &= 3\lambda y^2 \\ 3y^2 z &= 12 \end{aligned}$$

The second equation gives $y = 0$ (which does not work in the last equation) or $\lambda = \frac{8}{y}$. Using $\lambda = \frac{8}{y}$, the first equation becomes $6y+24z = 48z$, i.e., $y = 4z$. Substituting this last expression into $3y^2z = 12$ gives $48z^3 = 12$, i.e., $z = \sqrt[3]{\frac{1}{4}} = \frac{\sqrt[3]{16}}{4}$. Then $y = 4z = \sqrt[3]{16} = 2\sqrt[3]{2}$ and $x = 3y = 6\sqrt[3]{2}$. This is the only "critical point", and since $C(x, y, z)$ clearly goes to ∞ as any one of x, y, or z approaches zero or ∞, it must be an absolut minimum. So the dimensions that minimize the cost are $x = 6\sqrt[3]{2}, y = 2\sqrt[3]{2}$, and $z = \frac{\sqrt[3]{16}}{4}$.

10. Compute the iterated integral

$$\int_0^1 \int_{\tan^{-1} x}^{\pi/4} x dy dx.$$

11. Let D_1 be the region bounded by the polar curve $r(\theta) = R + \cos\theta$, where $R \geq 1$.

    (a) Compute the area of D_1.
    Solution:

$$\begin{aligned} A(D_1) &= \int_0^{2\pi} \int_0^{R+\cos\theta} r dr d\theta \\ &= \int_0^{2\pi} \left(\frac{r^2}{2}\right)_0^{R+\cos\theta} d\theta \\ &= \tfrac{1}{2} \int_0^{2\pi} (R^2 + 2R\cos\theta + \cos^2\theta) d\theta \\ &= \tfrac{1}{2} \int_0^{2\pi} \left(R^2 + 2R\cos\theta + \tfrac{1+\cos 2\theta}{2}\right) d\theta \\ &= \tfrac{1}{2} \left(R^2\theta + 2R\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)_0^{2\pi} \\ &= \pi R^2 + \tfrac{\pi}{2} \end{aligned}$$

(b) Explain why D_1 is *not* a circle.

Solution: The polar curve $r(\theta) = R + \cos\theta$ is symmetric over the x-axis, with x-intercepts $(R+1, 0)$ and $(-R+1, 0)$. If it were a circle, the radius would then be $\frac{1}{2}(R+1-(-R+1)) = R$, and the area would be πR^2. Since the area is not πR^2, the curve cannot be a circle.

(c) Let D_2 be the circle $x^2 + y^2 = R^2$. Show that the ratio of the areas of D_1 and D_2 approaches 1 as R goes to infinity.

Solution: $\lim\limits_{R\to\infty} \frac{\pi R^2 + \pi/2}{\pi R^2} = \lim\limits_{R\to\infty} \frac{\pi + \pi/2R^2}{\pi} = 1$

12. Let S be the solid region in the first octant bounded by the surface $z = xy^2$, the cylinder $x^2 + y^2 = 4$, and the xy-plane.

(a) What is the highest point in the region S?

Solution: S is bounded above by the surface $z = xy^2$, below by the xy-plane, and on the side by the cylinder $x^2 + y^2 = 4$. We need to find the maximum value of $f(x, y) = xy^2$ on the region $R = \{(x, y) | x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$. Setting $\nabla f = \langle y^2, 2xy\rangle = \langle 0, 0\rangle$ gives $y = 0$, while x can be any real number. So the only critical points of f occur along the x-axis. But the x-axis is already part of the boundary of the region R, so the maximum must occur along the boundary. As since $f(x, y) = 0$ along both the x-axis and the y-axis, and $f(x, y) > 0$ if both x and y are positive, the maximum must occur on the circle $x^2 + y^2 = 4$. Using Lagrange Multipliers, we have

$$\begin{aligned} y^2 &= 2\lambda x \\ 2xy &= 2\lambda y \\ x^2 + y^2 &= 4. \end{aligned}$$

The second equation gives $y = 0$ or $\lambda = x$. Since we already know that the maximum does not occur where $y = 0$, we conclude that $\lambda = x$. The first equation then becomes $y^2 = 2x^2$, and when combining this with the third equation we get $3x^2 = 4$, i.e., $x = \frac{2\sqrt{3}}{3}$. So then $y = \frac{2\sqrt{6}}{3}$, and as this is the only "critical point" on the quarter circle $x^2 = y^2 = 4, x \geq 0, y \geq 0$, it must be the location of the highest point. So the highest point in S is $\left(\frac{2\sqrt{3}}{3}, \frac{2\sqrt{6}}{3}, \frac{16\sqrt{3}}{9}\right)$.

(b) Sketch S.

Solution: See part (a) for a description of the region S. The traces of the surface $z = xy^2$ in planes parallel to the xz-plane are lines, and in planes parallel to the yz-plane are parabolas.

(c) Compute the volume of S.

Solution:

$$\begin{aligned}
V(S) &= \int\int_R xy^2 dA \\
&= \int_0^{\pi/2} \int_0^2 r^4 \sin^2\theta \cos\theta dr d\theta \\
&= \int_0^2 r^4 dr \int_0^{\pi/2} \sin^2\theta \cos\theta d\theta \\
&= \left(\frac{r^5}{5}\right)_0^2 \left(\frac{\sin^3\theta}{3}\right)_0^{\pi/2} \\
&= \left(\frac{32}{5}\right)\left(\frac{1}{3}\right) \\
&= \frac{32}{15}
\end{aligned}$$

13. Compute the volume of the solid in the first octant bounded by the planes $2x + y + 3z = 6, 2x - y = 0, x = 0$, and $z = 0$.