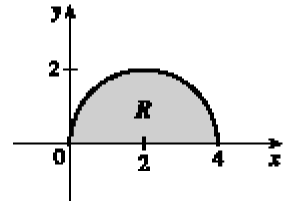
12.4: 8, 18, 24, 26

8. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 4\cos\theta,\ 0 \le \theta \le \pi/2\}$. Since $r = 4\cos\theta \iff r^2 = 4r\cos\theta \iff x^2+y^2 = 4x \iff (x-2)^2 + y^2 = 4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

18. The sphere $x^2+y^2+z^2 = 16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16} \sqrt{16-x^2-y^2}\,dA \qquad \text{[by symmetry]}$$

$$= 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta\int_2^4 r(16-r^2)^{1/2}dr$$

$$= 2\big[\,\theta\,\big]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0-12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

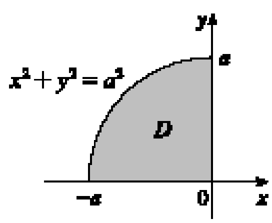
24. $D = \{(r,\theta) \mid 0\le\theta\le 2\pi,\ 0\le r\le 4+3\cos\theta\}$, so

$$A(D) = \iint_D dA = \int_0^{2\pi}\int_0^{4+3\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta} d\theta = \tfrac{1}{2}\int_0^{2\pi}(4+3\cos\theta)^2 d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}(16+24\cos\theta+9\cos^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\left(16+24\cos\theta+9\cdot\tfrac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \tfrac{1}{2}\left[16\theta+24\sin\theta+\tfrac{9}{2}\theta+\tfrac{9}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{41}{2}\pi$$

26.



$$\int_{\pi/2}^{\pi}\int_0^a (r\cos\theta)^2(r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{\pi}\int_0^a r^4\cos^2\theta\sin\theta\,dr\,d\theta$$

$$= \int_{\pi/2}^{\pi}\cos^2\theta\sin\theta\,d\theta\int_0^a r^4\,dr$$

$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_{\pi/2}^{\pi}\left[\tfrac{1}{5}r^5\right]_0^a$$

$$= -\tfrac{1}{3}\left(\cos^3\pi - \cos^3\tfrac{\pi}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{15}a^5$$