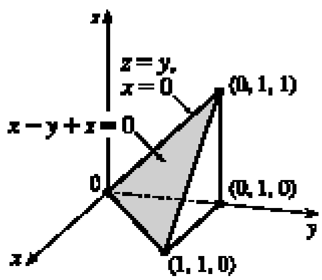
12.7: 8, 12, 20, 32, 38

8. $\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1\int_0^x \left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$

$$= \frac{1}{2}\int_0^1\int_0^x 3x^2y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1 \left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$$

$$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$$

12.



$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \frac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

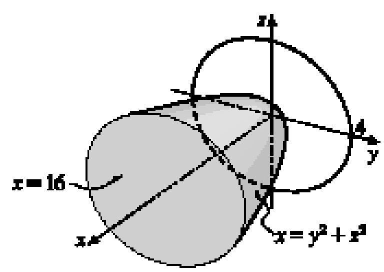
$$= \frac{1}{24}\int_0^1 y^4\,dy = \frac{1}{24}\left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{120}$$

20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.

Thus, $E = \{(x, y, z) \mid y^2 + z^2 \le x \le 16, y^2 + z^2 \le 16\}$.

Let $D = \{(y, z) \mid y^2 + z^2 \le 16\}$. Then using polar coordinates

$y = r\cos\theta$ and $z = r\sin\theta$, we have

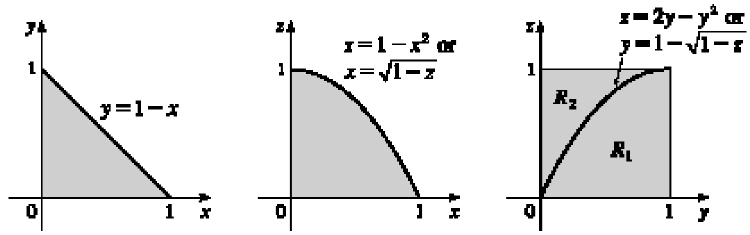


$$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2 + z^2))\,dA$$

$$= \int_0^{2\pi}\int_0^4 (16 - r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3)\,dr$$

$$= \left[\theta\right]_0^{2\pi} \left[8r^2 - \frac{1}{4}r^4\right]_0^4 = 2\pi(128 - 64) = 128\pi$$

32.



The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

$$= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y,z) in R_1, $0 \le x \le 1 - y$ and for (y,z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

38. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$$= \int_0^1\left[\tfrac{1}{2}(1-x)^3 - \tfrac{1}{3}(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1(1-x)^3\,dx = \tfrac{1}{24}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y - xy^2\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{2}x(1-x)^3 - \tfrac{1}{3}x(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1\left(x - 3x^2 + 3x^3 - x^4\right)dx = \tfrac{1}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right) = \tfrac{1}{120}$$

$$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{3}(1-x)^4 - \tfrac{1}{4}(1-x)^4\right]dx = \tfrac{1}{12}\left[-\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{60}$$

$$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\tfrac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$= \tfrac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx = \tfrac{1}{2}\int_0^1\left[\tfrac{1}{2}(1-x)^4 - \tfrac{2}{3}(1-x)^4 + \tfrac{1}{4}(1-x)^4\right]dx$$

$$= \tfrac{1}{24}\int_0^1(1-x)^4\,dx = -\tfrac{1}{24}\left[\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{120}$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\tfrac{1}{5},\tfrac{2}{5},\tfrac{1}{5}\right)$.