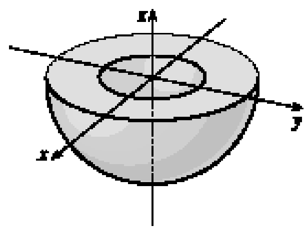
12.8: 4, 16, 28, 30(a), 34

4.



The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi \, d\phi \int_1^2 \rho^2 \, d\rho$$

$$= \big[\,\theta\,\big]_0^{2\pi} \big[-\cos\phi\big]_{\pi/2}^{\pi} \big[\tfrac{1}{3}\rho^3\big]_1^2$$

$$= 2\pi(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}$$

16. Since density is proportional to the distance from the z-axis, we can say $\rho(x, y, z) = K\sqrt{x^2 + y^2}$. Then

$$m = 2\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2 - r^2}} K r^2 \, dz \, dr \, d\theta = 2K \int_0^{2\pi} \int_0^a r^2 \sqrt{a^2 - r^2} \, dr \, d\theta$$

$$= 2K \int_0^{2\pi} \left[\tfrac{1}{8} r(2r^2 - a^2)\sqrt{a^2 - r^2} + \tfrac{1}{8} a^4 \sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta = 2K \int_0^{2\pi} \left[\left(\tfrac{1}{8}a^4\right)\left(\tfrac{\pi}{2}\right)\right] d\theta = \tfrac{1}{4} a^4 \pi^2 K$$

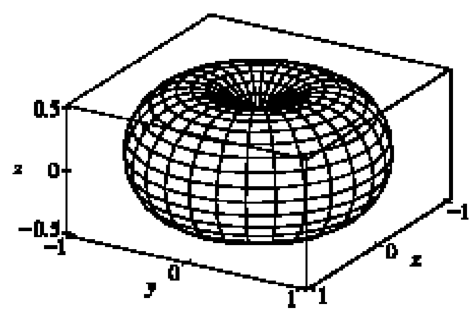
28. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$$V = \int_0^{\pi/6} \int_0^{\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a \rho^2 \, d\rho = \tfrac{\pi}{6}(2)\left(\tfrac{1}{3}a^3\right) = \tfrac{1}{9}\pi a^3.$$

30. (a) The region enclosed by the torus is $\{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le \sin\phi\}$, so its volume is

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \tfrac{1}{3} \sin^4\phi \, d\phi = \tfrac{2}{3}\pi \left[\tfrac{3}{8}\phi - \tfrac{1}{4}\sin 2\phi + \tfrac{1}{16}\sin 4\phi\right]_0^{\pi} = \tfrac{1}{4}\pi^2.$$

(b) In Maple, we can plot the torus using the `plots[sphereplot]` command, or with the `coords=spherical` option in a regular `plot` command. In Mathematica, use `ParametricPlot3d`.



34. The region of integration is the solid sphere $x^2 + y^2 + z^2 \le a^2$, so $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $0 \le \rho \le a$. Also $x^2 z + y^2 z + z^3 = (x^2 + y^2 + z^2)z = \rho^2 z = \rho^3 \cos\phi$, so the integral becomes

$$\int_0^{\pi} \int_0^{2\pi} \int_0^a \left(\rho^3 \cos\phi\right) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi} \sin\phi \cos\phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^5 \, d\rho = \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi} \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{6}\rho^6\right]_0^a = 0$$